Risk Quantification and Allocation Methods for Practitioners

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Risk Quantification and Allocation Methods for Practitioners

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Preface

This book aims to provide a broad introduction to quantification issues of risk management. The main function of the book is to present concepts and techniques in the assessment of risk and the forms that the aggregate risk may be distributed between business units. The book is the result of our research projects and professional collaborations with the financial and insurance sectors over last years. The textbook is intended to give a set of technical tools to assist industry practitioners to take decisions in their professional environments. We assume that the reader is familiar with financial and actuarial mathematics and statistics at graduate level.
This book is structured in two parts to facilitate reading: (I) Risk assessment, and (II) Capital allocation problems. Part (I) is dedicated to investigate risk measures and the implicit risk attitude in the choice of a particular risk measure, from a quantitative point of view. Part (II) is devoted to provide an overview on capital allocation problems and to highlight quantitative methods and techniques to deal with these problems. Illustrative examples of quantitative analysis are developed in the programming language R. Examples are devised to reflect some real problems that practitioners must frequently face in the financial or the insurance sectors. A collection of complementary material to the book is available in `http://www.ub.edu/rfa/R/`

Part (I) covers from Chapters 1 to 5. With respect to risk measures, it seemed adequate to deepen in the advantages and pitfalls of most commonly used risk measures in the actuarial and financial sectors, because the discussion could result attractive both to practitioners and supervisor authorities. This perspective allows to list some of the additional proposals that can be found in the academic literature and, even, to devise a family of alternatives called GlueVaR. Chapters in this part are structured as follows:

Chapter 1 - Preliminary concepts on quantitative risk measurement

This chapter contains some preliminary comments, notations and definitions related to quantitative risk assessment to keep the book as self-contained as possible.

Chapter 2 - Data on losses for risk evaluation

A descriptive statistical analysis of the dataset used to illustrate risk measurement and allocation in each chapter of the book is here presented.

Chapter 3 - A family of distortion risk measures

A new family of risk measures, called GlueVaR, is defined within the class of distortion risk measures. The relationship between GlueVaR, Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR) is explained. The property of subadditivity is investigated for GlueVaR risk measures, and the concavity in an interval of their associated distortion functions is analyzed.

Chapter 4 - GlueVaR and other new risk measures

This chapter is devoted to the estimation of GlueVaR risk values. Analytical closed-form expressions of GlueVaR risk measures are shown for the most frequently used distribution functions in financial and insurance applications, as well as Cornish-Fisher approximations for general skewed distribution functions. In addition, relationships between GlueVaR, Tail Distortion risk measures and RVaR risk measures are shown to close this chapter.

Chapter 5 - Risk measure choice

Understanding the risk attitude that is implicit in a risk assessment is crucial for decision makers. This chapter is intended to characterize the underlying risk attitude involved in the choice of a risk measure, when it belongs to the family of distortion risk measures. The concepts *aggregate risk attitude* and *local risk attitude* are defined and, once in hand, used to discuss the rationale behind choosing one risk measure or another among a set of different available GlueVaR risk measures in a particular example.

Part (II) covers from Chapters 6 to 8. Capital allocation problems fall on the disaggregation side of risk management. These problems are associated to a wide variety of periodical management tasks inside the entities. In an insurance firm, for instance, risk capital allocation by business lines is a fundamental element for decision making from a risk management point of view. A sound implementation of capital allocation techniques may help insurance companies to improve their underwriting risk and to adjust the pricing of their policies, so to increase the value of the firm. Chapters in this part are structured as follows:

Chapter 6 - An overview on capital allocation problems

There is a strong relationship between risk measures and capital allocation problems. Briefly speaking, most solutions to a capital allocation problem are determined by selecting one allocation criterion and choosing a particular risk measure. This chapter is intended to detect additional key elements involved in a solution to a capital allocation problem, in order to obtain a detailed initial picture on risk capital allocation proposals that can be found in the academic literature.
Personal notations and points of view are stated here and used from this point forward. Additionally, some particular solutions of interest are commented, trying to highlight both advantages and drawbacks of each one of them.

Chapter 7 - Capital allocation based on GlueVaR

This chapter is devoted to show how GlueVaR risk measures can be used to solve problems of proportional capital allocation through an example. Two proportional capital allocation principles based on GlueVaR risk measures are defined and an example is presented, in which allocation solutions with particular GlueVaR risk measures are discussed and compared with the solutions obtained when using the rest of alternatives.

Chapter 8 - Capital allocation principles as compositional data

In the last chapter, some connections between capital allocation problems and aggregation functions are emphasized. The approach is based on functions and operations defined in the standard simplex which, to the best of our knowledge, remained an unexplored approach.

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PART I

RISK ASSESSMENT

1 Preliminary concepts on quantitative risk measurement

This chapter is structured in two parts. The first one is intended to compile a set of theoretical definitions that we consider useful and relevant for quantitative risk managers. These definitions are related to the quantitative risk assessment framework of unidimensional risk factors, so other key issues like multivariate dependence are not covered herein. In our opinion, the concepts addressed in this chapter are the building blocks of unidimensional risk measurement which need to be helpful to practitioners. A careful first reading of this part is not necessary if one is already familiar with the fundamental ideas, because our aim is to leave it as a reference point to which to go back whenever needed. The second part serves to introduce ideas to bear in mind when moving from theory to practice. As before, this selection is subjective and it relies on our judgment, and the reader could consider the subjects in this selection too specific or too obvious. This is also the reason why we close the chapter with some brief remarks, in which we provide additional topics to be aware of and selected references in the literature to become an expert on risk quantification.

1.1 Risk measurement - Theory

1.1.1 First definitions

Definition 1.1 (Probability space). A probability space is defined by three elements $(\Omega, \mathscr{A}, P)$. The sample space Ω is a set of all possible events of a random experiment, $\mathscr{A}$ is a family of the set of all subsets of Ω (denoted as $\mathscr{A} \in \wp(\Omega)$) with a σ-algebra structure, and the probability P is a mapping from $\mathscr{A}$ to $[0,1]$ such that $P(\Omega) = 1$, $P(\varnothing) = 0$ and P satisfies the σ-additivity property.

Some remarks regarding the previous definition. $\mathscr{A}$ has a σ-algebra structure if $\Omega \in \mathscr{A}$, if $A \in \mathscr{A}$ implies that $\Omega \smallsetminus A = A^c \in \mathscr{A}$ and if $\bigcup_{n\geqslant 1} A_n \in \mathscr{A}$ for any numerable set $\{A_n\}_{n\geqslant 1}$. Additionally, the σ-additivity property aforementioned states that if $\{A_n\}_{n\geqslant 1}$ is a succession of pairwise disjoint sets belonging to $\mathscr{A}$ then

$$P\left(\bigcup_{n=1}^{+\infty} A_n\right) = \sum_{n=1}^{+\infty} P(A_n).$$

A probability space is finite, i.e. $\Omega = \{\varpi_1, \varpi_2, \ldots, \varpi_n\}$, if the sample space is finite. Then $\wp(\Omega)$ is the σ-algebra, which is denoted as 2^{Ω}. In the rest of this book, N instead of Ω and m instead of ϖ are used when referring to finite probability spaces. Hence, the notation is $\left(N, 2^N, P\right)$, where $N = \{m_1, m_2, \ldots, m_n\}$.

Definition 1.2 (Random variable). Let $(\Omega, \mathscr{A}, P)$ be a probability space. A random variable X is a mapping from Ω to $\mathbb{R}$ such that $X^{-1}((-\infty, x]) := \{\varpi \in \Omega : X(\varpi) \leqslant x\} \in \mathscr{A}, \forall\, x \in \mathbb{R}$.

A random variable X is discrete if $X(\Omega)$ is a finite set or a numerable set without cumulative points.

Definition 1.3 (Distribution function of a random variable). Let X be a random variable. The distribution function of X, denoted by F_X, is defined by $F_X(x) := P\left(X^{-1}((-\infty, x])\right)$. The notation $P(X \leqslant x) = P\left(X^{-1}\ ((-\infty, x])\right)$ is commonly used, so expression $F_X(x) = P(X \leqslant x)$ is habitual. The distribution function of a random variable is also known as the cumulative distribution function (cdf) of that random variable.

The distribution function F_X is non-decreasing, right-continuous and satisfies that $\lim\limits_{x\to-\infty} F_X(x) = 0$ and $\lim\limits_{x\to+\infty} F_X(x) = 1$.

Definition 1.4 (Survival function of a random variable). Let X be a random variable. The survival function of X, denoted by S_X, is defined by $S_X(x) := P\left(X^{-1}\left((x, +\infty)\right)\right)$. The following notation is commonly used, $P(X > x) = P\left(X^{-1}\left((x, +\infty)\right)\right)$, so expression $S_X(x) = P(X > x)$ is habitual. So, the survival function S_X can be expressed as $S_X(x) = 1 - F_X(x)$, for all $x \in \mathbb{R}$.

The survival function S_X is non-increasing, left-continuous and satisfies that $\lim\limits_{x\to-\infty} S_X(x) = 1$ and $\lim\limits_{x\to+\infty} S_X(x) = 0$. Note that the domain of the distribution function and the survival function is $\mathbb{R}$ even if X is a discrete random variable. In other words, F_X and S_X are defined for $X(\Omega) = \{x_1, x_2, \ldots, x_n, \ldots\}$ but also for any $x \in \mathbb{R}$.

Definition 1.5 (Density function). A function f defined from $\mathbb{R}$ to $\mathbb{R}$ is a *density function* if $f \geqslant 0$, if it is Riemann integrable in $\mathbb{R}$ and if the following equality holds:

$$\int_{-\infty}^{+\infty} f(t)dt = 1.$$

A random variable X is absolutely continuous with density f_X if its distribution function F_X can be written as $F_X(x) = \int_{-\infty}^{x} f_X(t)dt$ for all $x \in \mathbb{R}$. Let us remark that, in such a case, the derivative function of F_X is f_X, so $dF_X(x) = f_X(x)$.
If X is a discrete random variable such that $X(\Omega) = \{x_1, x_2, \ldots, x_n, \ldots\}$ then for if $x \in \{x_1, x_2, \ldots, x_n, \ldots\}$, the density function may be defined by $f_X(x) = P(X = x_i)$ and $f_X(x) = 0$ if $x \notin \{x_1, x_2, \ldots, x_n, \ldots\}$.
Apart from discrete and absolutely continuous random variables there are random variables that are not discrete neither absolutely continuous but belong to a more general class. These random variables are such that their distribution function satisfies that

$$F_X(x) = (1-p)\cdot F_X^c(x) + p \cdot F_X^d(x) \tag{1.1}$$

for a certain $p \in (0,1)$, and where F_X^c is a distribution function linked to an absolutely continuous random variable and F_X^d is a distribution function associated to a discrete random variable X^d with $X^d(\Omega) = \{x_1, x_2, \ldots, x_n, \ldots\}$.

Definition 1.6 (Mathematical expectation). Three different cases are considered in this definition.

Discrete case

Let X be a discrete random variable with $X(\Omega) = \{x_1, x_2, \ldots, x_n, \ldots\}$. X has finite expectation if $\sum_{i=1}^{+\infty} |x_i| \cdot P(X = x_i) < +\infty$. If this condition is satisfied then the mathematical expectation of X is $\mathbb{E}(X) \in \mathbb{R}$, where $\mathbb{E}(X)$ is defined by

$$\mathbb{E}(X) = \sum_{i=1}^{+\infty} x_i \cdot P(X = x_i) = \sum_{i=1}^{+\infty} x_i \cdot f_X(x_i).$$

Absolutely continuous case

Let X be an absolutely continuous random variable with density function f_X. X has finite expectation if $\int_{-\infty}^{+\infty} |x| \cdot f_X(x)dx < +\infty$. If this condition is

Table 1.1 Examples of random variables

Type of r.v.	Name of r.v.	Distribution function
Discrete	Binomial, $X \sim B(m, q)$	$F_X(x) = \sum_{k \leqslant x} \binom{m}{k} \cdot q^k \cdot (1-q)^{m-k}$
Absolutely continuous	Normal, $X \sim N(\mu, \sigma^2)$	$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left\{-\frac{1}{2\sigma^2} \cdot (t-\mu)^2\right\} dt$
Mixed	Mixed exponential	$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-(1-p)\cdot\exp\{-\lambda \cdot x\} & \text{if } x \geqslant 0 \end{cases}$
	The probability of $\{X = 0\}$ is equal to $p \in (0,1)$, the probability of $\{X < 0\}$ is zero and strictly positive values have assigned a probability of and exponential r.v. of parameter $\lambda > 0$, additionally multiplied by $1 - p$.	

satisfied then the mathematical expectation of X is $\mathbb{E}(X) \in \mathbb{R}$, where $\mathbb{E}(X)$ is defined by

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} |x| \cdot f_X(x)dx < +\infty.$$

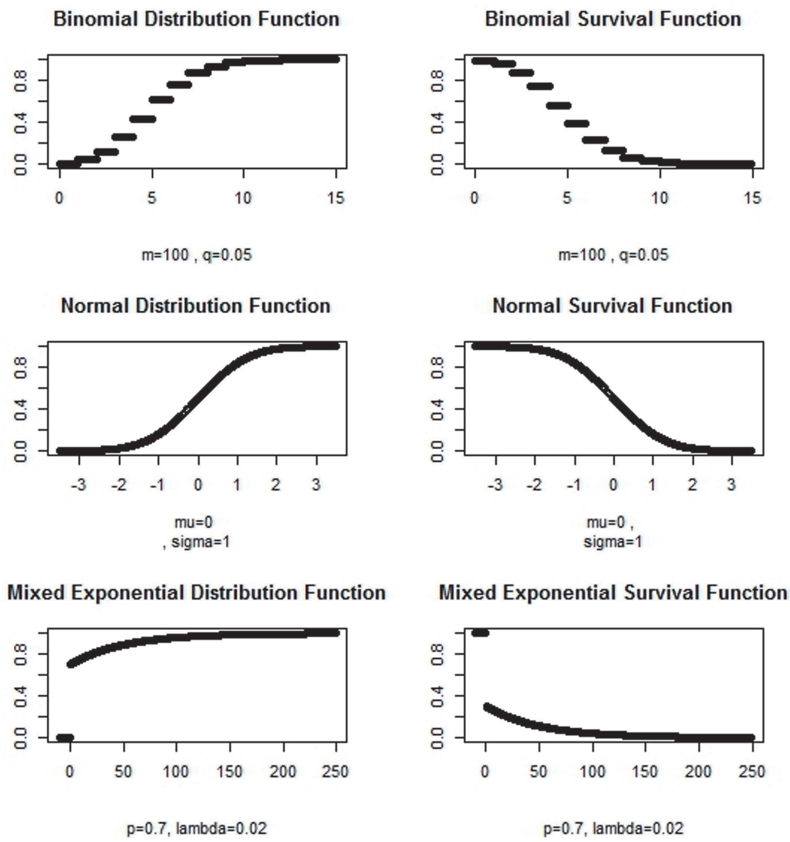
General case

Let X be a random variable with distribution function of the form (1.1), and such that

$$\begin{cases} p \cdot F_X^d(x) = \sum_{x_i \leqslant x} \left(F_X(x_i) - \lim_{t \to x_i,\, t < x_i} F_X(t)\right) = \sum_{x_i \leqslant x} P(X = x_i), \\ (1-p) \cdot F_X^c(x) = F_X(x) - p \cdot F_X^d(x) = \int_{-\infty}^{x} f_X^c(t)dt, \end{cases}$$

where $\{x_1, x_2, \ldots, x_n, \ldots\}$ is the set of discontinuity points of F_X. In this case, if the random variables linked to F_X^d and F_X^c respectively have finite expec-

Figure 1.1 Graphs of distribution and survival functions of random variables from Table 1.1, with parameters $m = 100$, $q = 5\%$, $\mu = 0$, $\sigma = 1$, $p = 70\%$ and $\lambda = 0.02$.



tation then

$$\mathbb{E}(X) = \sum_{i=1}^{+\infty} x_i \cdot P(X = x_i) + \int_{-\infty}^{+\infty} x \cdot f_X^c(x)\,dx.$$

Note that the differential function of a distribution function F_X, which will be denoted dF_X and is usually known as probability density function (pdf), may be defined by

$$dF_x(x) = \begin{cases} P(X = x_i) & \text{if } x \in \{x_1, x_2, \ldots, x_n, \ldots\}, \\ f_X^c(x) & \text{if } x \notin \{x_1, x_2, \ldots, x_n, \ldots\}, \end{cases} \tag{1.2}$$

Taking advantage of this notation, if the random variables involved have finite expectation then the mathematical expectation in the discrete, the absolutely continuous or the general cases can always be written as

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x \cdot dF_X(x).$$

This expression unifies the ones used in Definition 1.6 and makes further reading easier than more complicated notation.
The following result will be really helpful in several parts of this book, although comments on its usefulness cannot be provided at this stage. The result shows how to interpret the mathematical expectation of a random variable in terms of its survival function.

Proposition 1.1. *Let X be a random variable with finite expectation. The following equality holds:*

$$\mathbb{E}(X) = \int_{-\infty}^{0} (S_X(t) - 1)dt + \int_{0}^{+\infty} S_X(t)dt. \tag{1.3}$$

Proof. Each summand in (1.3) is treated separately, despite the idea behind the proof is basically the same. First of all, consider

$$a = \int_{-\infty}^{0} (S_X(t) - 1)dt \text{ and } b = \int_{0}^{+\infty} S_X(t)dt.$$

With this notation, $\mathbb{E}(X) = a + b$ has to be proved. In order to prove that, let us recall that $\mathbb{E}(X) = \int_{-\infty}^{+\infty} x \cdot dF_X(x)$ and rewrite this last expression as

$$\mathbb{E}(X) = \int_{-\infty}^{0} x \cdot dF_X(x) + \int_{0}^{+\infty} x \cdot dF_X(x) = a' + b'.$$

Using Fubini's theorem in $(*)$:

$$\begin{aligned} b' &= \int_{0}^{+\infty} x \cdot dF_X(x) = \int_{0}^{+\infty} \left(\int_{t=0}^{x} dt \right) dF_X(x) \\ &\overset{(*)}{=} \int_{t=0}^{+\infty} \left(\int_{x=t}^{+\infty} dF_X(x) \right) dt = \int_{t=0}^{+\infty} (F_X(+\infty) - F_X(t))dt \\ &= \int_{t=0}^{+\infty} (1 - F_X(t))dt = \int_{0}^{+\infty} S_X(t)dt \\ &= b. \end{aligned}$$

$$
\begin{aligned}
a' &= \int_{-\infty}^{0} x \cdot dF_X(x) = \int_{x=-\infty}^{0} \left(\int_{t=0}^{x} dt \right) dF_X(x) \\
&\overset{(*)}{=} \int_{t=-\infty}^{0} \left(\int_{x=-\infty}^{t} (-dF_X(x)) \right) dt = \int_{t=-\infty}^{0} \left(\int_{x=-\infty}^{t} (dS_X(x)) \right) dt \\
&= \int_{t=-\infty}^{0} (S_X(t) - S_X(-\infty)) dt \\
&= \int_{t=-\infty}^{0} (S_X(t) - 1) dt \\
&= a.
\end{aligned}
$$

□

The proposition has been proved, using that $F_X(+\infty) = \lim_{x\to+\infty} F_X(x) = 1$, $S_X(-\infty) = \lim_{x\to-\infty} S_X(x) = \lim_{x\to-\infty}(1 - F_X(x)) = 1 - \lim_{x\to-\infty} F_X(x) = 1$ and $dS_X(x) = d[1 - F_X(x)] = -dF_X(x)$.

Definition 1.7 (Risk measure). Let Γ be the set of all random variables defined for a given probability space $(\Omega, \mathscr{A}, P)$. A risk measure is a mapping ρ from Γ to $\mathbb{R}$, so $\rho(X)$ is a real value for each $X \in \Gamma$.

Frequently, the set Γ is considered to be the set of p-measurable functions defined on the probability space, $p \geqslant 0$. In other words, frequently $\Gamma = \mathscr{L}^p\{(\Omega, \mathscr{A}, P)\}$. For more details see, for instance, Rüschendorf [2013] and the references therein.
The most frequently used, or well known, risk measures in the insurance and financial industry are listed in next paragraph. It has to be noted that insurance and financial perspectives may differ in some aspects. Detailed comments on these differences are provided in Section 1.2. Our perspective is the actuarial one and, hence, the following definitions are aligned with this point of view. In fact, these definitions are basically taken from Denuit *et al.* [2005]. The reason of including these definitions is to avoid possible misunderstandings due to differences in names given to certain risk measures.

Definition 1.8 (Value at Risk). Let us consider $\alpha \in (0,1)$. The function

$$
\begin{aligned}
\mathrm{VaR}_\alpha : \Gamma &\longrightarrow \mathbb{R} \\
X &\longmapsto \mathrm{VaR}_\alpha(X) = \inf\{x \mid F_X(x) \geqslant \alpha\}
\end{aligned}
$$

is a risk measure called *Value at Risk at confidence level* α. If F_X is continuous and strictly increasing then $\mathrm{VaR}_\alpha(X) = F_X^{-1}(X)$, where F_X^{-1} is the inverse of the distribution function of random variable X.

Definition 1.9 (Tail Value at Risk). Let us consider $\alpha \in (0, 1)$. The function

$$\begin{aligned} \mathrm{TVaR}_\alpha : \Gamma &\longrightarrow \mathbb{R} \\ X &\longmapsto \mathrm{TVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\lambda(X)\,d\lambda \end{aligned}$$

is a risk measure called *Tail Value at Risk at confidence level* α.

Definition 1.10 (Conditional Tail Expectation). Let us consider $\alpha \in (0, 1)$. The function

$$\begin{aligned} \mathrm{CTE}_\alpha : \Gamma &\longrightarrow \mathbb{R} \\ X &\longmapsto \mathrm{CTE}_\alpha(X) = \mathbb{E}\left[X \mid X > \mathrm{VaR}_\alpha(X)\right] \end{aligned}$$

is a risk measure called *Conditional Tail Expectation at confidence level* α.

Definition 1.11 (Conditional Value at Risk). Let us consider $\alpha \in (0, 1)$. The function

$$\begin{aligned} \mathrm{CVaR}_\alpha : \Gamma &\longrightarrow \mathbb{R} \\ X &\longmapsto \mathrm{CVaR}_\alpha(X) = \mathbb{E}\left[X - \mathrm{VaR}_\alpha(X) \mid X > \mathrm{VaR}_\alpha(X)\right] \\ &\qquad = \mathrm{CTE}_\alpha(X) - \mathrm{VaR}_\alpha(X) \end{aligned}$$

is a risk measure called *Conditional Value at Risk* at confidence level α.

Definition 1.12 (Expected Shortfall). Let be $\alpha \in (0, 1)$. The function

$$\begin{aligned} \mathrm{ES}_\alpha : \Gamma &\longrightarrow \mathbb{R} \\ X &\longmapsto \mathrm{ES}_\alpha(X) = \mathbb{E}\left[(X - \mathrm{VaR}_\alpha(X))_+\right] \end{aligned}$$

is a risk measure called *Expected Shortfall at confidence level* α. Notation $(t)_+$ is used to refer to the function that returns 0 if $t \leqslant 0$ and t otherwise.

The following relationships between previous risk measures hold, as stated in Section 2.4 of Denuit *et al.* [2005]:

$$\mathrm{TVaR}_\alpha(X) = \mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha} \cdot \mathrm{ES}_\alpha(X), \tag{1.4}$$

$$\mathrm{CTE}_\alpha(X) = \mathrm{VaR}_\alpha(X) + \frac{1}{S_X(\mathrm{VaR}_\alpha(X))} \cdot \mathrm{ES}_\alpha(X), \tag{1.5}$$

$$\mathrm{CVaR}_\alpha(X) = \frac{\mathrm{ES}_\alpha(X)}{S_X(\mathrm{VaR}_\alpha(X))}. \tag{1.6}$$

Note that relationships (1.4) and (1.5) imply that, if the distribution function of random variable X is continuous and strictly increasing then $\mathrm{TVaR}_\alpha(X) = \mathrm{CTE}_\alpha(X)$ because

$$S_X(\mathrm{VaR}_\alpha(X)) = 1 - F_X(\mathrm{VaR}_\alpha(X)) = 1 - F_X\left(F_X^{-1}(\alpha)\right) = 1 - \alpha.$$

This is the reason of finding expressions like: 'roughly speaking, the TVaR is understood as the mathematical expectation beyond VaR' in this book.

Example 1.1 (Illustrative exercise). Let us consider the following random variable X, that measures a loss, i.e. an economic value that can be lost with a certain probability,

x_i	-100	0	50	200	500
$p_i = P(X = x_i)$	0.2	0.5	0.25	0.04	0.01

a) Calculate $\mathrm{VaR}_\alpha(X)$, $\mathrm{TVaR}_\alpha(X)$ and $\mathrm{CTE}_\alpha(X)$ for $\alpha = 90\%$ and for $\alpha = 99\%$.

b) Explain if a loss X which is distributed like in the table presented here can produce a TVaR at the 90% level that is equal to 180.

c) Find the value that must substitute 200 so that the results exactly correspond to $\mathrm{ES}_{90\%}(X) = 13$, for a confidence level equal to 90%. Verify also that if we replace value 200 by 250 and value 500 by 550, then we obtain again the same results for a confidence level equal to 90%.

d) Based on the ideas in step c), explain why the value of the risk measures do not determine in a unique way the distribution of a random loss.

Solution **a)** In order to make calculations easier, we complete the initial table with two additional rows. One corresponds to the distribution function (cdf) of random variable X and the other is the corresponding survival function.

x_i	-100	0	50	200	500
$p_i = P(X = x_i)$	0.2	0.5	0.25	0.04	0.01
$F_X(x_i)$	0.2	0.7	0.95	0.99	1
$S_X(x_i)$	0.8	0.3	0.05	0.01	0

We calculate the values of $\mathrm{VaR}_{90\%}(X)$ and $\mathrm{VaR}_{99\%}(X)$ using Definition 1.8 $\mathrm{VaR}_{\alpha}(X)$and the information displayed on the table. So,

$$\mathrm{VaR}_{90\%}(X) = \inf\{x \mid F_X(x) \geqslant 90\%\} = 50\,,$$
$$\mathrm{VaR}_{99\%}(X) = \inf\{x \mid F_X(x) \geqslant 99\%\} = 200\,.$$

Both for the calculation of TVaR and CTE, we need to obtain the value of ES beforehand. Let us remind the definition of the latter for a loss random variable X and a confidence level $\alpha \in (0,1)$:

$$\mathrm{ES}_{\alpha}(X) = \mathbb{E}\left[(X - \mathrm{VaR}_{\alpha}(X))_{+}\right]\,.$$

Note that we need to consider $Z_{X,\alpha} = (X - \mathrm{VaR}_{\alpha}(X))_{+}$, which is equal to zero when $x_i - \mathrm{VaR}_{\alpha}(X) \leqslant 0$ and which is equal to $x_i - \mathrm{VaR}_{\alpha}(X)$ when the difference is positive. Let us add two more lines to the table that has been used in this exercise, corresponding to values $Z_{X,90\%}$ and $Z_{X,99\%}$:

x_i	-100	0	50	200	500
$p_i = P(X = x_i)$	0.2	0.5	0.25	0.04	0.01
$F_X(x_i)$	0.2	0.7	0.95	0.99	1
$S_X(x_i)$	0.8	0.3	0.05	0.01	0
$(x_i - 50)_{+}$	0	0	0	150	450
$(x_i - 200)_{+}$	0	0	0	0	300

Therefore,

$$\mathrm{ES}_{90\%}(X) = \sum_{i=1}^{5} (x_i - 50)_{+} \cdot p_i = 150 \cdot 0.04 + 450 \cdot 0.01 = 6 + 4.5 = 10.5,$$
$$\mathrm{ES}_{99\%}(X) = \sum_{i=1}^{5} (x_i - 200)_{+} \cdot p_i = 300 \cdot 0.01 = 3.$$

Once the values for ES are obtained, then we can calculate TVaR and CTE using the following expressions:

$$\mathrm{TVaR}_{\alpha}(X) = \mathrm{VaR}_{\alpha}(X) + \frac{1}{1-\alpha} \cdot \mathrm{ES}_{\alpha}(X)$$

and

$$\mathrm{CTE}_{\alpha}(X) = \mathrm{VaR}_{\alpha}(X) + \frac{1}{S_X(\mathrm{VaR}_{\alpha}(X))} \cdot \mathrm{ES}_{\alpha}(X).$$

$$\begin{aligned}
\mathrm{TVaR}_{90\%}(X) &= 50 + (1/0.1)10.5 = 155\,, \\
\mathrm{TVaR}_{99\%}(X) &= 200 + (1/0.01)3 = 500\,; \text{ and} \\
\mathrm{CTE}_{90\%}(X) &= 50 + (1/0.05)10.5 = 260\,, \\
\mathrm{CTE}_{99\%}(X) &= 200 + (1/0.01)3 = 500\,.
\end{aligned}$$

b) The random loss X that is considered in this exercise cannot correspond to another loss if some values of the risk measures at the confidence level of 90% are different to the risk measures obtained for the loss. For example, if the TVaR at the 90% level is 180 while we just saw that TVaR at the confidence level of 90% is 155 for the loss in this exercise, then the two random variables differ in their distribution.

c) Let us fix the level of confidence to 90%. Let us note in that case that the source of the difference between the risk measures TVaR and CTE in two cases is in the value of $\mathrm{ES}_{90\%}(X)$. For instance if the value is 13, while it is 10.5 in section a) of the current exercise. Then, when looking at the calculation of $\mathrm{ES}_{90\%}(X)$, what needs to be done is to look at the following equation:

$$(x_4 - 50)\cdot 0.04 + 450\cdot 0.01 = 13, \quad \text{with} \quad x_4 \geqslant 50.$$

Then, solving the previous equation, we obtain

$$x_4 = 25\cdot[13 - 4.5 + 2] = 25\cdot[10.5] = 262.5.$$

Furthermore, if we change $x_4 = 200$ by $x_4 = 262.5$ we obtain the results that we were aiming at, namely,

$$\mathrm{VaR}_{90\%}(X) = 50, \quad \mathrm{ES}_{90\%}(X) = 13, \quad \mathrm{CVaR}_{90\%}(X) = 260,$$
$$\mathrm{TVaR}_{90\%}(X) = 180, \quad \text{and} \quad \mathrm{CTE}_{90\%}(X) = 310.$$

The variant proposed here is to consider now that x_4 equals 250 and x_5 equals 550, and leaving all other x_i as they were initially set. So, the value of $\mathrm{ES}_{90\%}(X)$ is calculated as

$$\begin{aligned}
(250 - 50)_+ \cdot 0.04 + (550 - 50)_+ \cdot 0.01 &= 200\cdot 0.04 + 500\cdot 0.01 \\
&= 8 + 5 = 13.
\end{aligned}$$

Therefore, with this change, we obtain

$$\mathrm{VaR}_{90\%}(X) = 50, \quad \mathrm{ES}_{90\%}(X) = 13, \quad \mathrm{CVaR}_{90\%}(X) = 260,$$
$$\mathrm{TVaR}_{90\%}(X) = 180 \quad \text{and} \quad \mathrm{CTE}_{90\%}(X) = 310.$$

d) In the previous paragraph, we deduce that at least, there are two random losses that have the same values for

$$\mathrm{VaR}_{90\%}(X), \quad \mathrm{ES}_{90\%}(X), \quad \mathrm{CVaR}_{90\%}(X),$$
$$\mathrm{TVaR}_{90\%}(X) \quad \text{and} \quad \mathrm{CTE}_{90\%}(X).$$

As a consequence, we have just seen that the values of the risk measures do not determine in a unique fashion the cumulative probability function for a random variable.

1.1.2 Properties for risk measures

A list of properties that a risk measure may or may not satisfy is presented herein. Most of these properties have an economic interpretation or, at least, a relationship with some features that practitioners (the ones who want to quantify risk) demand to the risk measure (the instrument to quantify risk). In order to summarize the properties and their interpretation, Table 1.2 is provided.

Table 1.2 Properties for risk measures

Property	Idea behind the property
Translation invariance $\rho(X+c) = \rho(X) + c$, $\forall\, c \in \mathbb{R}$	If a positive non random quantity c is added to random loss X then it is required to the risk measure that the risk value of the new loss should be increased by the same quantity. Otherwise, if the quantity c is negative (so a protection buffer has been added to the original random loss X) then the risk measure should reflect this buffer as a net effect on the original risk value.
Subadditivity $\rho(X_1 + X_2) \leqslant \rho(X_1) + \rho(X_2)$	If a risk measure satisfies this property then it is able to quantitatively reflect the idea that diversification is a strategy that does not increase risk.

Continued on next page

Table 1.2: continued from previous page

Property	Idea behind the property
Monotonicity $P(X_1 \leqslant X_2) = 1 \Rightarrow \rho(X_1) \leqslant \rho(X_2)$	If losses of a position are almost surely worse than losses of another position, then the risk value of the former should be greater than the risk value of the latter.
Positive homogeneity $\rho(c \cdot X) = c \cdot \rho(X)$, $\forall\, c > 0$	If losses to which the risk manager is exposed are multiples of a particular loss, then it is required that the risk measure of the overall risk should be the same multiple of the risk value of that particular loss.
Comonotonic additivity X_1 and X_2 comonotonic $\Rightarrow \rho(X_1 + X_2) = \rho(X_1) + \rho(X_2)$	Informally, two random variables are comonotonic if they are linked to another random variable that drives their behavior. This property is intended to identify those risk measures that take into account this underlying relationship between comonotonic random variables and, as a consequence, they do not assign quantitative diversification benefits when considering the sum of those random variables.
Convexity $\rho(\lambda \cdot X_1 + (1-\lambda) \cdot X_2) \leqslant \lambda \cdot \rho(X_1) + (1-\lambda) \cdot \rho(X_2)$, $\forall\, \lambda \in (0,1)$	This is a sort of generalization of the subadditivity property. If the risk figure of any linear combination of two random variables is smaller than the associated linear combination of risk figures, then the risk measure captures diversification benefits in a continuous way. Note that if the risk measure is convex and positively homogeneous and considering $X_i' = 2 \cdot X_i$ and $\lambda = 1/2$, then the subadditivity property for X_i', $i = 1, 2$ is obtained.

Continued on next page

Table 1.2: continued from previous page

Property	Idea behind the property
Law invariance (objectivity) If $P(X_1 \leqslant x) = P(X_2 \leqslant x), \forall\, x \in \mathbb{R}$ then $\rho(X_1) = \rho(X_2)$	If two random variables have identical distribution functions then it is required to the risk measure that their risk values should be identical too.
Relevance If $X \geqslant 0$ and $X \neq 0$ then $\rho(X) > 0$	If a random loss is not zero then its risk value should be strictly positive.
Strictness $\rho(X) \geqslant \mathbb{E}(X)$	This property is intended to detect those risk measures that are conservative enough to be used as a management tool, in other words, risk values based in risk measures that satisfy this property are always greater that the expected loss.

For any random variables $X_1, X_2, X \in \Gamma$.

Financial and actuarial literature are plenty of interesting proposals of risk measures. Details on some of these proposals are provided in Chapters 3 and 4 and, in addition, several other references are pointed out therein.

1.2 Risk measurement - Practice

Let us start this section with Table 1.3, in which closed-form expressions are provided for VaR and TVaR where random variable X is distributed as a Normal ($\mathscr{N}$), a Lognormal ($\mathscr{LN}$) and a Generalized Pareto ($\mathscr{GP}$) distribution. Notation conventions are used. Namely, ϕ and Φ stand for the standard Normal pdf and cdf, respectively. The standard Normal distribution α-quantile is denoted as $q_\alpha = \Phi^{-1}(\alpha)$. For the $\mathscr{GP}$ distribution, the definition provided in Hosking and Wallis [1987] is considered, where the scale parameter is denoted by σ and k is the shape parameter. The $\mathscr{GP}$ distribution contains the Uniform ($k = 1$), the Exponential ($k = 0$), the Pareto ($k < 0$) and the type II Pareto ($k > 0$) distributions as special cases. Table 1.3 is basically taken from Sandström [2011].

Table 1.3 Analytical closed-form expressions of VaR and TVaR for selected random variables

Random variable	Risk measure	Expression
Normal distribution $\mathcal{N}(\mu,\sigma^2)$	VaR_α	$\mu+\sigma\cdot q_\alpha$
	TVaR_α	$\mu+\sigma\cdot\dfrac{\phi(q_\alpha)}{1-\alpha}$
Lognormal distribution $\mathcal{LN}(\mu,\sigma^2)$	VaR_α	$\exp(\mu+\sigma\cdot q_\alpha)$
	TVaR_α	$\exp\left(\mu+\dfrac{\sigma^2}{2}\right)\cdot\dfrac{\Phi(\sigma-q_\alpha)}{1-\alpha}$
Generalized Pareto distribution $\mathcal{GPD}(0,\sigma)$ (Exponential distribution)	VaR_α	$-\sigma\cdot\ln(1-\alpha)$
	TVaR_α	$\sigma\cdot[1-\ln(1-\alpha)]$
Generalized Pareto distribution $\mathcal{GPD}(k,\sigma)$ with $k<0$	VaR_α	$\dfrac{\sigma}{k}\left[1-(1-\alpha)^k\right]$
	TVaR_α	$\begin{cases} +\infty & \text{if } k\leqslant -1 \\ \dfrac{\sigma}{k}\left[1-(1-\alpha)^k\right]+\dfrac{\sigma}{k}\left[\dfrac{k\cdot(1-\alpha)^k}{k+1}\right] & \text{if } k\in(-1,0) \end{cases}$

1.2.1 'Liability side' versus 'asset side' perspectives

No matter if you come from the insurance or from the financial industry: in both cases you agree on thinking on risk in terms of random losses. Differences arise when quantifying risk in practice, because usually an actuary works with random variables in which positive values identify losses and, therefore, she is worried about what happens in the right tail of the distributions. Nonetheless, a practitioner from the financial industry usually works with random variables where positive values identify gains or profits, so she is mainly worried about the behavior of the left tail of the distributions. Therefore, depending on where you come from, you would be used to look at risk quantification from different perspectives. More precisely, we should talk about 'liability side' practitioners and 'asset side' practitioners

instead of 'insurance' and 'financial' practitioners. For instance, an example of financial practitioners that take (what we have called) a 'liability side' perspective when quantifying risk are those in charge of assessing credit risk. On the other side, as we will discuss later, the perspective used in European insurance regulation to quantify solvency capital requirements is an 'asset side' perspective and not a 'liability side' perspective (as it could be expected because of the nature of this industry's business).
Although moving from one perspective to the other is not a big issue, few guidelines to reach this goal are outlined. It is our opinion that these are the kind of helpful indications that bridge the gap between theory and practice, and between insurance ('liability side') and financial ('asset side') practitioners. The following guidelines are summarized in Table 1.4, in order to provide a fast and visual reference when needed.

Table 1.4 Risk quantification: 'liability side' versus 'asset side' perspectives

Concept	Liability side perspective	Asset side perspective
Notation for risk measures used in this Table	ρ	r
Target random variable	X a random loss	X a random profit
Monotonicity	$P(X_1 \leqslant X_2) = 1 \Rightarrow \rho(X_1) \leqslant \rho(X_2)$	$P(X_1 \leqslant X_2) = 1 \Rightarrow \mathrm{r}(X_1) \geqslant \mathrm{r}(X_2)$
	From the liability side perspective, smaller losses should have associated smaller risk measurements. On the asset side perspective, the higher the gain the lesser its risk value.	
Translation invariance	$\rho(X+c) = \rho(X) + c, \forall c \in \mathbb{R}$	$\mathrm{r}(X+c) = \mathrm{r}(X) - c, \forall c \in \mathbb{R}$

Continued on next page

Table 1.4: continued from previous page

Concept	**Liability side perspective**	**Asset side perspective**
	A positive amount of money from the liability side perspective may be considered as a loss, while from the asset side perspective it is exactly the opposite. Therefore, if the risk measure satisfies the translation invariance property, a positive amount of money must increase risk from the liability side perspective while the same positive amount of money must decrease risk from the asset side perspective.	
Relevance	$X \geqslant 0$ and $X \neq 0 \Rightarrow \rho(X) > 0$	$X \leqslant 0$ and $X \neq 0 \Rightarrow \mathrm{r}(X) > 0$
Strictness	$\rho(X) \geqslant \mathbb{E}(X)$	$\mathrm{r}(X) \geqslant -\mathbb{E}(X)$
	Recalling that X represents a random loss from the liability side perspective and a gain from the asset side perspective.	
Subadditivity, Positive homogeneity, Comonotonic additivity, Convexity, Law invariance	Formal expressions from both perspectives remain as they are displayed in Table 1.2, except for replacing ρ by r.	
For any random variables $X_1,\ X_2, X \in \Gamma$.		

Additional comments with respect to differences among the 'liability side' and the 'asset side' perspective for risk quantification may be found, for instance, in Rüschendorf [2013]. As an example, Definition 1.8 has been introduced from a 'liability side' perspective, so positive values of random variable X are considered losses. Considering expressions in Definition 1.8 and adopting an 'asset side' perspective, if one is interested in obtaining the VaR at α confidence level for a continuous random variable Z with positive values representing profits, then the correct risk figure would be obtained as

$$\begin{aligned} &\text{'VaR of } Z \text{ at confidence level } \alpha \in (0,1)\text{'} \\ &\qquad = -\mathrm{VaR}_{1-\alpha}(Z) \text{ following Definition 1.8.} \end{aligned} \tag{1.7}$$

The perspective taken in the following chapters of this book is the one that we have called 'liability side' perspective.

1.2.2 Some misunderstandings to be avoided in practice

Risk measures versus their estimates

It is quite frequent to confuse a risk measure with the procedures used to estimate it. These two concepts are different and their identification can lead to misunderstandings. Fortunately, the spread of knowledge about risk measurement makes these kind of doubts less frequent than they were before. But when having first contact with risk measurement (for instance, if you are an undergraduate student interested in this topic or a recently hired practitioner without previous experience in the insurance industry or the financial sector) this is one of the most common mistakes. Diagram in Figure 1.2 may help to clarify concepts.

Figure 1.2 Basic mind map for risk quantification.

$$\begin{tikzcd} \textbf{Theory} \arrow[r, "\text{Assumptions}"] & \textbf{Practice} \\ \text{Risk measure } (\rho) \arrow[r, "\rho \text{ To } \widehat{\rho}"] & \boxed{\text{Risk figure est. } \widehat{\rho}(\widehat{X})} \\ \text{Random variable } (X) \arrow[r, "X \text{ To } \widehat{X}"] & \text{R.v. estimation } (\widehat{X}) \arrow[u] \end{tikzcd} \tag{1.8}$$

Figure 1.2 is intended to depict a schematic situation faced when trying to quantify risk. On the one hand, theoretical aspects related to the risk measure (the instrument to summarize risk) and the target random variable (the source of risk) must be taken into account. These theoretical aspects are represented on the left hand side of the diagram, and should correspond to answers to questions such as the following: Is the selected risk measure adequate? Is the target random variable observable?...On the other hand, figures are basic in practice. As long as the final objective is to obtain an estimate of the incurred risk (framed box in Figure 1.2) assumptions have to be in place to move from theory to practice. So, the assumptions made to estimate both the risk measure and the target random variable become crucial. They are so relevant that, from our point of view, they can lead to the confusion that we are highlighting here. This is because, in daily practice, one could deliver risk figures estimations (right hand side of the diagram) without worrying about theoretical aspects (left hand side). As mentioned before, let us put some examples.

Example 1.2 (Historical VaR). Measuring risk in practice using the historical VaR methodology has been relatively common because it has an easy implementation. Properly speaking, it is not a unique methodology as we try to justify hereinafter. From the point of view provided by the diagram in Figure 1.2, on the theoretical side this methodology takes into account as risk measure ρ the VaR with some confidence level $\alpha \in (0,1)$ and considers that the target random variable X is observable. Moreover, it is assumed that observations of that random variable from past periods can be obtained. The assumptions for moving from theory to practice are as follows: with respect to the estimation of the target random variable $\widehat{X}$, it is assumed that future realizations will be exactly the same as past realizations, so past observations that have been obtained are going to be considered future observations too. And with respect to the estimation $\widehat{\rho}$ of VaR, there is not a unique feasible assumption (and this is why we consider the 'historical VaR' a set of methodologies and not just one). For instance, a feasible assumption is to consider the data set of observations of $\widehat{X}$ as it represents the discrete random variable X which only takes those particular values and no more. Consequently, VaR should be estimated as the empirical α-quantile of that set. But, if the data set of observations of $\widehat{X}$ is considered just a sample of X, then any α-quantile approximation[1] of data set $\widehat{X}$ could be used to obtain the final risk figure estimation $\widehat{\rho}(\widehat{X})$ of $\rho(X)$.

Example 1.3 (Normal VaR). Bearing in mind diagram in Figure 1.2, this methodology takes as theoretical risk measure ρ the VaR at some confidence level $\alpha \in (0,1)$, and considers as target random variable X one which is assumed to be normally distributed. Assumptions to move from the theoretical side to the practical one are as follows: with respect to X, it is assumed

[1] For instance, `quantile` function in software R has more than 10 different ways to approximate the α-quantile, where the one coded by 0 is what we have called the empirical quantile. Even MS Excel has implemented functions `INC.PERCENTILE` and `EXC.PERCENTILE` which return different approximations of the α-quantile.

that $X \sim N(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$ and $\sigma > 0$, and that the practitioner is able to estimate μ and σ in some way (maybe from data or from expert judgment, for instance), so it is feasible to obtain $\widehat{\mu}$ and $\widehat{\sigma}$ estimates of μ and σ, respectively. With respect to the risk measure, the assumption made on the random variable implicitly provides a closed-form expression for VaR, because if $X \sim N(\mu, \sigma^2)$ then $\mathrm{VaR}_\alpha(X) = \mu + \sigma \cdot q_\alpha$, where q_α is the α-quantile of a standard normal distribution (as it has been shown in Table 1.3). As it happened with the historical VaR methodology, the Normal VaR methodology may be understood as a set of methodologies depending on the particular chosen way for estimating the parameters of the distribution. In the end, $\rho(X)$ is estimated by $\widehat{\mu} + \widehat{\sigma} \cdot q_\alpha$.

Note that the Normal VaR methodology is frequently used for sums of normally distributed random variables. On the theoretical side, if $n > 1$ random variables $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, \dots, n$, are considered and $\Lambda = (\rho_{ij})_{i,j \in \{1,\dots,n\}}$ is the correlation matrix for pairs of those random variables, then it is known that

$$X = \sum_{i=1}^{n} X_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sigma^2\right),$$

where $\sigma^2 = \vec{\mu}' \cdot \Lambda \cdot \vec{\mu}$ and $\vec{\mu}$ is an n-dimensional vector whose components are μ_i, $i = 1, \dots, n$. So, the situation is just the one described in the previous paragraph, taking as $\mu = \sum_{i=1}^{n} \mu_i$ and as $\sigma = \sqrt{\vec{\mu}' \cdot \Lambda \cdot \vec{\mu}}$. In this case, the process to obtain parameter estimates $\widehat{\mu}$ and $\widehat{\sigma}$ must take into account that correlation coefficients ρ_{ij} should also be estimated. In other words,

$$\widehat{\sigma} = \sqrt{\widehat{\vec{\mu}}' \cdot \widehat{\Lambda} \cdot \widehat{\vec{\mu}}}.$$

Example 1.4 (Cornish-Fisher VaR). As in the previous examples, different methodologies are embraced under this name. They share the following elements: on the one hand, the theoretical risk measure ρ is the VaR with some confidence level $\alpha \in (0, 1)$ and no hypothesis about the distribution function of the target random variable is made. Nonetheless, it is assumed that some higher order moments of X exist and are finite. On the other hand, assumptions for moving from the theoretical side to the practical side are that, in order to obtain an estimation $\widehat{\rho}(\widehat{X})$, a closed-form approximation similar to the one valid for normally distributed random variables is achievable. For that purpose, modified α-quantiles are devised taking into account estimations of finite order moments of X. Differences between Cornish-Fisher VaR methodologies come from the maximum order of moments considered in the quantile estimations. For instance, in Chapter 4 we have used third order Cornish-Fisher VaR approximations, but is is usual to find fourth order Cornish-Fisher VaR approximations in financial applications.

VaR versus Mean-VaR

An apparently harmless sentence like 'most financial credit risk models used in practice to quantify risk are based on VaR at some confidence level', which most practitioners and researchers in this field may subscribe, can have undesired consequences if it is misunderstood. The main concern with the previous sentence is that nothing is said about the random variable to which the VaR is applied to: even considering the same confidence level and the same input data, different figures can be obtained depending on the underlying random variable under inspection. For instance, a large number of banks use internal models to simulate losses generated by credit events affecting their loans. Let us focus on one bank and let us denote its aggregate simulated losses by X. Therefore, the amount of money needed to cover unexpected losses (its *economic capital*) is probably computed as

$$\mathrm{EC} = \mathrm{VaR}_{99.9\%}(X - \mathbb{E}(X))$$

in order to take into account its simulated values and also regulatory requirements (Basel II/III). Note that in this case, although the random variable simulated is X, the one used to quantify risk (i.e., to obtain the economic capital) is $U = X - \mathbb{E}(X)$, in fact. The VaR is a risk measure that satisfies the translation invariance property shown in Table 1.2 and, therefore,

$$\mathrm{EC} = \mathrm{VaR}_{99.9\%}(U) = \mathrm{VaR}_{99.9\%}(X) - \mathbb{E}(X). \tag{1.9}$$

This last expression for the EC is certainly more familiar to financial practitioners. Moreover, sometimes $\rho(X) = \mathrm{VaR}_{99.9\%}(X) - \mathbb{E}(X)$ is considered the value that another risk measure ρ named 'Mean Value at Risk'(Mean-VaR) returns when applied to random loss X. Expression (1.9) has been intentionally displayed in second place in order to stress the following idea. Let us imagine now an European insurance company calculating its Solvency Capital Requirement (SCR) under the Solvency II regulatory framework and by using an internal model. Let us suppose that within the model a set of stochastic basic own funds of the company for the next year is simulated. In such a case, if Y denotes the 'basic own funds for the next year' random variable, then taking into account expression (1.7) it seems reasonable that the following expression

$$\mathrm{SCR} = \mathrm{VaR}_{99.5\%}(-Y) = -\mathrm{VaR}_{0.5\%}(Y) \tag{1.10}$$

would be used to compute the SCR, because it perfectly fits the regulatory requirements[2]. But what it is relevant here is that it makes no sense to require the company to set aside, as a cushion against insolvency, the following amount of money

$$\mathrm{SCR} = \mathrm{VaR}_{99.5\%}(-Y) - \mathbb{E}(-Y) = \mathrm{VaR}_{99.5\%}(-Y) + \mathbb{E}(Y). \qquad (1.11)$$

Due to misunderstanding of expression (1.9) for the EC, and transposing it for the SCR expression simply replacing X by $-Y$, figures with non economic sense are attained. Why? Basically because X and $-Y$ are essentially different. Random variable X is a pure loss while $-Y$ contains both losses and gains. In fact, hopefully $\mathbb{E}(-Y) \ll 0$ (the insurance company expects substantial gains) and reasonably $\mathbb{E}(X) > 0$ (the expectation of a set of losses is also a loss). In words, when computing the EC the focus is set on random variable $U = X - \mathbb{E}(X)$ because it is assumed that the quantity $\mathbb{E}(X)$ is already accounted for on the liability side of the balance sheet (which is not entirely simulated by the credit risk model) to mitigate credit losses. On the other hand, the model for the SCR of the insurance company is simulating the whole balance sheet. Therefore $-Y$ is not comparable with X because losses associated to $-Y$ are those that have exceeded all the mitigation tools and strategies that the company has in place, while X losses are computed gross of any mitigation effect.

Example 1.5. A toy example can help us to illustrate the impact of such a misunderstanding. Imagine two insurance companies c_1 and c_2, one with $Y_{1,t} = 100$ monetary units (m.u.) of present basic own funds and the other with $Y_{2,t} = 1$ m.u. Both use the same model to project next year basic own funds (let us say $Y_{1,t+1}$ and $Y_{2,t+1}$) and the same methodology to compute VaR at the 99.5% confidence level. To simplify things, let us assume that $\mathbb{E}(Y_{i,t+1}) = Y_{i,t}$ for $i = 1, 2$, so the expectation of projected basic owns funds for the next year is nothing but the value of the actual basic own funds of each company. Imagine that the risk figures that these companies obtain are $\mathrm{VaR}_{99.5\%}(-Y_{1,t+1}) = 5$ and $\mathrm{VaR}_{99.5\%}(-Y_{2,t+1}) = 0.5$. They may be interpreted in the following way: c_1 is going to suffer a minimum loss of a 5% of its present basic own funds in a 0.5% of the future scenarios considered, while c_2 is going to suffer a minimum loss of a 50% of its present basic own funds in a 0.5% of the future scenarios considered. Interpreted in that way,

[2] As it is shown with this expression, the core of the European insurance regulation uses what we have called an 'asset side' perspective when talking about risk quantification.

c_2 seems highly riskier than c_1. And this would properly be reflected using expression (1.10), because their respective solvency capital requirements will be $\text{SCR}(c_1) = 5$ m.u. and $\text{SCR}(c_2) = 0.5$ m.u. which, in terms of their present basic own funds, represent reasonable risk proportions. But note that if misunderstandings are in place and expression (1.11) is used instead of expression (1.10) to compute their SCR, then $\text{SCR}(c_1) = 5 + 100 = 105$ m.u. and $\text{SCR}(c_2) = 0.5 + 1 = 1.5$ m.u. are obtained. These figures are far from representing neither the risk faced by the companies nor their relative riskiness.

Somebody could think that the previous examples overweight the importance of items on the right hand side of Figure 1.2. These examples have been chosen because they correspond to common risk quantification issues found in practice and researchers must bear them in mind. Nevertheless, it is also our intention to aware that practitioners should spend some time on thinking of questions related to the left hand side of that Figure, this is, on theoretical aspects related to a practical risk quantification in a regular basis. Some of these questions are listed below, although it is neither an extensive list nor a prioritized one:

- Have several risk measures been considered before the final selection is made?
- Do these risk measures satisfy properties that we consider necessary?
- Are these risk measures or their confidence levels regulatory driven?
- Have we an idea about the implicit risk attitude behind using those particular risk measures?
- What are we looking for as the final result of this risk quantification process?
- Are we aware about our capability (in terms of time, resources and knowledge) to transform ideas into numbers? In other words, for every considered risk measure and every target random variable, do we know how to move from the theoretical side to the practical side?
- Have we properly defined our target random variable?
- Does the target random variable depend on other random variables easier to measure or identify?
- How precise do we need to be in our estimations?

Hopefully, useful ideas about how to answer some of these question may be found in this book or in the references therein. We would like to close this chapter with some last remarks. As it has already been said, main references used to build this chapter are books Denuit *et al.* [2005] and Rüschendorf [2013]. Note that the CTE risk measure introduced in Definition 1.10 is called Expected Shortfall (ES) in McNeil *et al.* [2005] and, therefore, there is also a difference with the Definition 1.12 of ES provided in this book. Moreover, names for several risk measures in Section 1.1 do not match the ones used for equivalent risk measures in Rüschendorf [2013]. This remark makes evident that there is yet no common consensus in risk measures naming.

For an interesting way to study basic risk measures but without a parametric model assumption, the work by Alemany *et al.* [2013] shows how to implement kernel estimation of the probability density function and how to derive the risk measure from there. Kernel estimation is specially useful when the number of observations is large. Bolancé *et al.* [2003]; Buch-Larsen *et al.* [2005]; Bolancé *et al.* [2008] explain how to address heavy-tailed or skewed distributions. The interested reader can find several contributions using other models and non-parametric approaches in Bolance *et al.* [2008]; Guillen *et al.* [2011, 2013]. Bolancé *et al.* [2012, 2013] provide data-driven examples with R and SAS code in the context of operational risk problems. Multivariate risk quantification is addressed by Bolancé *et al.* [2014]; Bahraoui *et al.* [2014].

With respect to a deeper analysis of issues of Solvency II for practitioners and regarding theoretical aspects behind Cornish-Fisher expansions, the interested reader is referred to Sandström [2011]. Last but not least, one topic not covered by this book that has to be taken into account in risk quantification is the model risk. Aggarwal *et al.* [2016] provides a wide variety of approaches to deal with this real challenge and may be an interesting departure point to anyone interested in this topic.

1.3 Exercises

1. Consider the following empirical distribution

$$13,\ 15,\ 26,\ 26,\ 26,\ 37,\ 37,\ 100$$

Determine the $\mathrm{VaR}_{85\%}(X)$ and $\mathrm{TVaR}_{85\%}(X)$.

2. Consider the following distribution function $F(x) = \dfrac{x^2}{9}$ for $0 < x \leqslant 3$. Find the $\mathrm{VaR}_{85\%}(X)$ and $\mathrm{TVaR}_{85\%}(X)$.

3. Given that

$$\mathrm{VaR}_{90\%}(X) = 50, \quad \mathrm{ES}_{90\%}(X) = 13 \quad \text{and} \quad \mathrm{CVaR}_{90\%}(X) = 260.$$

a) Calculate $\mathrm{TVaR}_{90\%}(X)$, $S_X(\mathrm{VaR}_{90\%}(X))$ and $\mathrm{CTE}_{90\%}(X)$.

b) Discuss if it is possible that loss X would be an absolutely continuous random variable.

4. Show that the TVaR of a random variable X distributed by the Normal distribution $\mathcal{N}(\mu, \sigma^2)$ is equal to $\mathrm{TVaR}_\alpha = \mu + \sigma \cdot \dfrac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha}$, where ϕ and Φ^{-1} stand for the standard Normal pdf and quantile function, respectively.

a) Demonstrate that the properties of *Translation invariance*, *Positive homogeneity* and *Strictness* are satisfied in this case.

b) Repeat the exercise for the CVaR_α.

5. Analyze if the properties of *Translation invariance*, *Positive homogeneity* and *Strictness* are satisfied by the VaR and TVaR when:

a) the random variable X is distributed by the Lognormal distribution $\mathcal{LN}(\mu, \sigma^2)$.

b) the random variable X is distributed by the Generalized Pareto distribution $\mathcal{GPD}(0, \sigma)$.