Geometric Stability Conditions on Finite Free Abelian Quotients

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Bridgeland Stability Conditions

Definition

A **numerical Bridgeland stability condition** on a triangulated category $\mathcal{D}$ is a pair: $\sigma = (\mathcal{A}, Z)$ such that

- $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$.
- $Z \colon \mathrm{K}(\mathcal{A}) \to \mathbf{C}$ is a group homomorphism such that for all $0 \neq A \in \mathcal{A}$, $Z([A]) \in \mathbf{H}$. Z must also satisfy the *Harder-Narasimhan property* and factor via $\mathrm{K}_{\mathrm{num}}(\mathcal{A})$.

Moreover, σ must satisfy the *support property*.

Theorem (Bridgeland '07)

The set of all numerical Bridgeland stability conditions on a triangulated category $\mathcal{D}$, denoted $\mathsf{Stab}(\mathcal{D})$, *has the structure of a complex manifold.*

Let X be a smooth complex projective variety, and let $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(X)$

Definition

$\sigma \in \mathsf{Stab}(X) := \mathsf{Stab}(\mathrm{D}^{\mathrm{b}}(X))$ is called **geometric** if $\mathcal{O}_x$ is σ-stable for all $x \in X$.

Geometric stability conditions

Definition

$\sigma \in \mathrm{Stab}(X)$ is called **geometric** if $\mathcal{O}_x$ is σ-stable for all $x \in X$.

What is known?

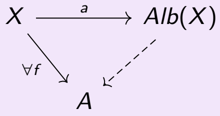
- $\dim X = 1$: $\mathrm{Stab}^{\mathrm{Geo}}(X) \cong \mathbf{C} \times \mathbf{H}$
- $\dim X = 2$: General construction produces geometric stability conditions
- $\dim X = 3$: General construction for some 3-folds produces geometric stability conditions
- $\dim X \geq 4$: **???**

Theorem (Lie Fu - Chunyi Li - Xiaolei Zhao '21)

If X has finite Albanese morphism, then $\mathrm{Stab}(X) = \mathrm{Stab}^{\mathrm{Geo}}(X)$

Question: What about the converse?

Albanese morphism



Every algebraic variety X has a map a to its Albanese variety, $Alb(X) := \mathrm{Pic}^0(\mathrm{Pic}^0(X))$. This map is called the **Albanese morphism** of X. It is algebraic, and every morphism $f : X \to A$ to another abelian variety A factors via a.

The Le Potier Function

Definition/Proposition (D.)

Let X be a surface. Let $H, B \in \mathrm{NS}_{\mathbf{R}}(X) := \mathrm{NS}(X) \otimes \mathbf{R}$ with H ample. We define the Le Potier function twisted by B, $\Phi_{X,H,B} \colon \mathbf{R} \to \mathbf{R}$, as

$$\Phi_{X,H,B}(x) := \limsup_{\mu \to x} \left\{ \frac{\mathrm{ch}_2(F) - B\,.\,\mathrm{ch}_1(F)}{H^2 \mathrm{ch}_0(F)} : \begin{array}{l} F \in \mathrm{Coh}(X) \text{ is } H\text{-semistable} \\ \text{with } \mu_H(F) = \mu \end{array} \right\}.$$

Proposition (D.)

$\Phi_{X,H,B}$ is well defined and satisfies $\Phi_{X,H,B}(x) \leq \frac{1}{2}\left(x - \frac{H.B}{H^2}\right)^2 - \frac{B^2}{2H^2}$

Theorem (D.)

$\Phi_{X,H,B}$ controls $\mathrm{Stab}^{\mathrm{Geo}}(X)$. In particular:

$$\mathrm{Stab}^{\mathrm{Geo}}(X) \cong \mathbf{C} \times \left\{ (H, B, \alpha, \beta) \in \mathrm{NS}_{\mathbf{R}}(X)^2 \times \mathbf{R}^2 : H \text{ ample}, \alpha > \Phi_{X,H,B}(\beta) \right\}.$$

Fu, Li, and Zhao introduced the Le Potier function with $B = 0$.

1. They showed: If $\rho(X)=1$, then $\Phi_{X,H} := \Phi_{X,H,0}$ controls $\mathrm{Stab}^{\mathrm{Geo}}(X)$.
2. They conjectured: If $H^1(\mathcal{O}_X) = 0$, then $\Phi_{X,H}$ is discontinuous at 0.

Computations with $\Phi_{X,H}$

Proposition (D.)

Suppose G is a finite group acting freely on X, and let $\pi\colon X \to S := X/G$ be the quotient map. Let $H_X = \pi^* H_S$ be an ample class pulled back from S. Then $\Phi_{S,H_S}(x) = \Phi_{X,H_X}(x)$.

Proposition (D.)

Suppose X has finite Albanese morphism, and $H_X = a^* H_A$ be an ample class pulled back from $Alb(X)$. Then $\Phi_{X,H_X}(x) = \frac{x^2}{2}$.

Beauville-type Surfaces

Let $X = C_1 \times C_2$ where $g(C_i) > 0$. Suppose G is a finite abelian group acting freely on X such that $S := X/G$ satisfies $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$. These were classified by Bauer and Catanese in 2003. G is one of the following groups: $(\mathbf{Z}/2\mathbf{Z})^3$, $(\mathbf{Z}/2\mathbf{Z})^4$, $(\mathbf{Z}/3\mathbf{Z})^2$, $(\mathbf{Z}/5\mathbf{Z})^2$

X has finite Albanese morphism, but S does not. One can find $H \in \mathrm{Amp}_{\mathbf{R}}(X)$ that is pulled back from S and $Alb(X)$. By the above theorems, $\Phi_{S,H_S}(x) = \Phi_{X,H_X}(x) = \frac{x^2}{2}$. This disproves ②.

Group Actions on Triangulated Categories

Let G be a finite group, and let $\mathcal{D}$ be a triangulated category.

Definition (Deligne)

A **(right) action** of G on $\mathcal{D}$ is defined by:

- a functor $\phi_g \colon \mathcal{D} \to \mathcal{D}$, for every $g \in G$;
- a natural isomorphism $\varepsilon_{g,h} \colon \phi_g \phi_h \to \phi_{hg}$ for every $g, h \in G$, such that these isomorphisms are compatible with triples of group elements.

Deligne also defines $\mathcal{D}_G$, the category of G-equivariant objects

Examples

- Suppose G acts on X, define $\phi_g := g^* \colon \mathrm{D}^\mathrm{b}(X) \to \mathrm{D}^\mathrm{b}(X)$. There are canonical isomorphisms $g^* h^* \to (hg)^*$. Moreover, $(\mathrm{D}^\mathrm{b}(X))_G = \mathrm{D}^\mathrm{b}_G(X) \cong \mathrm{D}^\mathrm{b}([X/G])$
- If G is abelian and acts on $\mathcal{D}$, then $\widehat{G} := \mathrm{Hom}(G, \mathbf{C}^*)$ acts on $\mathcal{D}_G$.

Theorem (Elagin '15)

Let G be abelian. Then $(\mathcal{D}_G)_{\widehat{G}} \cong \mathcal{D}$.

Group Actions on Triangulated Categories Continued

Theorem (Macrí-Mehrotra-Stellari '09, D.)

There is a one-to-one correspondence:

$$\left\{ \begin{array}{c} G\text{-invariant stability} \\ \text{conditions on } \mathcal{D} \end{array} \right\} \underset{\mathrm{Forg}_{\widehat{G}}^{-1}}{\overset{\mathrm{Forg}_G^{-1}}{\rightleftarrows}} \left\{ \begin{array}{c} \widehat{G}\text{-invariant stability} \\ \text{conditions on } \mathcal{D}_G \end{array} \right\} \qquad (*)$$

One can show that geometric stability conditions are preserved under $(*)$. This can be used to deduce the following result:

Theorem (D.)

Let X be a surface with finite Albanese morphism. Let G be a finite abelian group acting freely on X and let $S := X/G$. Then $(\mathrm{Stab}(S))^{\widehat{G}} = \mathrm{Stab}^{\mathrm{Geo}}(S)$, and this is a connected component of $\mathrm{Stab}(S)$.

This disproves the expectation in the literature that there would exist a wall of the geometric chamber on S.

Any questions?

- Bridgeland, Tom. "Stability Conditions on Triangulated Categories." In: *Annals of Mathematics* 166.2 (2007): 317-345
- Deligne, Pierre. "Action du groupe des tresses sur une catégorie." In: *Inventiones mathematicae* 128.1 (1997): 159–175
- Elagin, Alexey. "On equivariant triangulated categories." *arXiv preprint arXiv:1403.7027* (2014).
- Fu, Lie, Li, Chunyi, and Zhao, Xiaolei. "Stability manifolds of varieties with finite Albanese morphisms." In: *Transactions of the American Mathematical Society* 375 (2022): 5669-5690.
- Bauer, Ingrid, Catanese, Fabrizio. "Some new surfaces with $p_g = q = 0$." *arXiv preprint arXiv:math/0310150* (2003)
- Macrì, Emanuele, Mehrotra, Sukhendu, and Stellari Paolo. "Inducing stability conditions." In: *Journal of Algebraic Geometry* 18.4 (2009): 605-649.