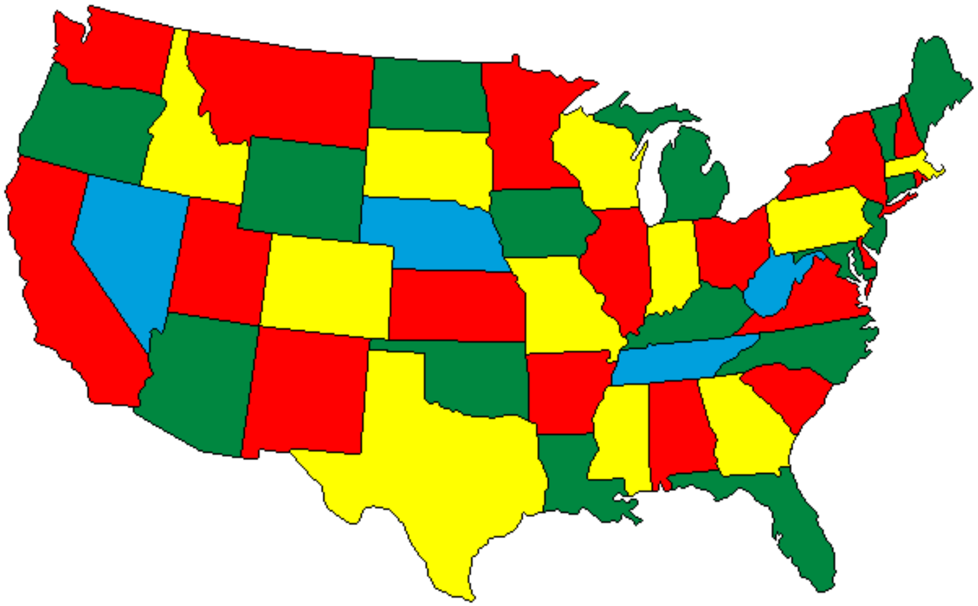
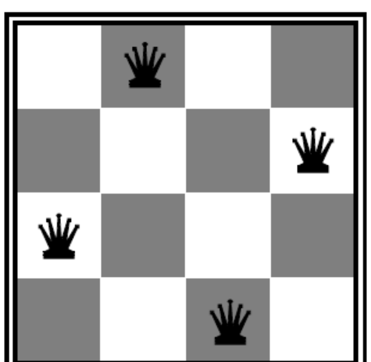
Constraint Satisfaction Problems (Chapter 6)





8			4		6			7
						4		
	1					6	5	
5		9		3		7	8	
				7				
	4	8		2		1		3
	5	2					9	
		1						
3			9		2			5

Two classes of search problems

- Assumptions: single agent, deterministic, fully observable, discrete environment
- **Search for *planning***
  - The path to the goal is the important thing
  - Paths have various costs, depths
- **Search for *assignment***
  - Assign values to variables while respecting certain constraints
  - The goal (complete, consistent assignment) is the important thing

8			4		6			7
						4		
	1					6	5	
5		9		3		7	8	
				7				
	4	8		2		1		3
	5	2					9	
		1						
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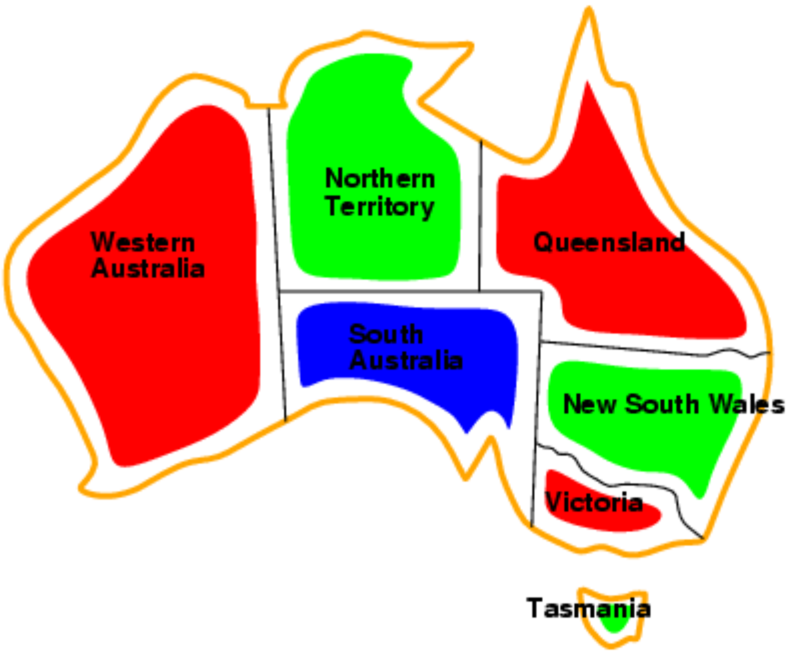
Constraint satisfaction problems (CSPs)

- Definition:
  - **State** is defined by variables X_i with values from domain D_i
  - **Goal test** is a set of constraints specifying allowable combinations of values for subsets of variables
  - **Solution** is a complete, consistent assignment
- How does this compare to the “generic” tree search formulation?
  - A more explicit representation for states and goal test
  - Allows for more efficient specialized search algorithms

Example: Map Coloring

- **Variables:** WA, NT, Q, NSW, V, SA, T
- **Domains:** {red, green, blue}
- **Constraints:** adjacent regions must have different colors e.g., WA ≠ NT, or (WA, NT) in {(red, green), (red, blue), (green, red), (green, blue), (blue, red), (blue, green)}

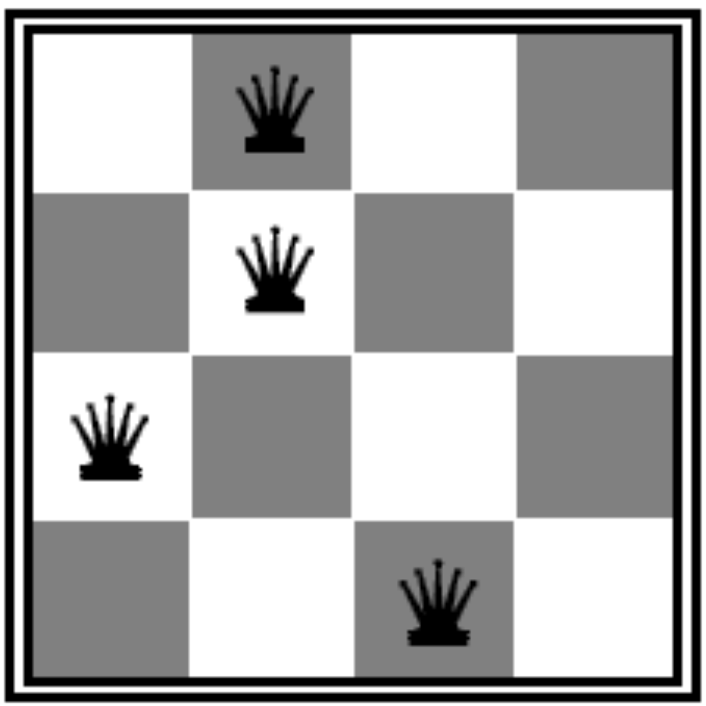
Example: Map Coloring

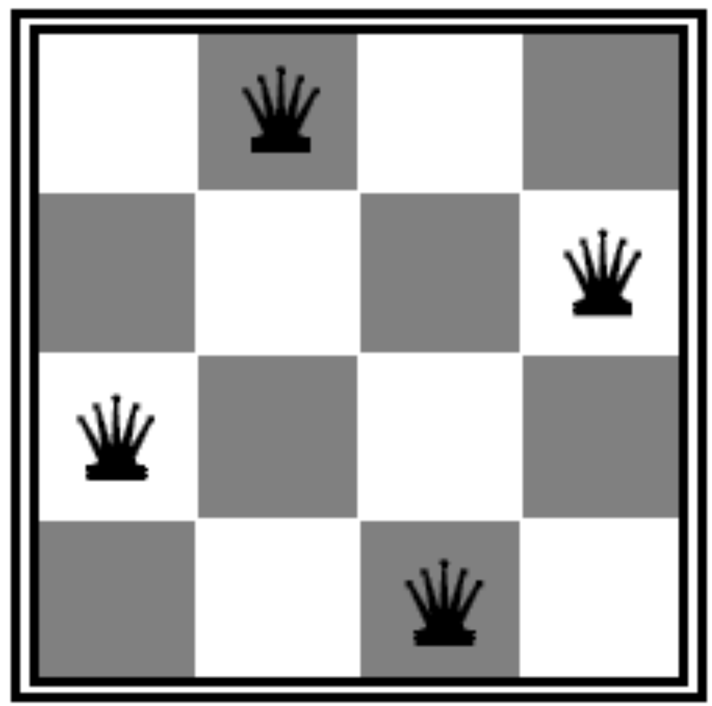


- **Solutions** are *complete* and *consistent* assignments, e.g., WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green

Example: *n*-queens problem

- Put n queens on an $n \times n$ board with no two queens on the same row, column, or diagonal





Example: N-Queens

- **Variables:** X_{ij}
- **Domains:** {0, 1}
- **Constraints:**

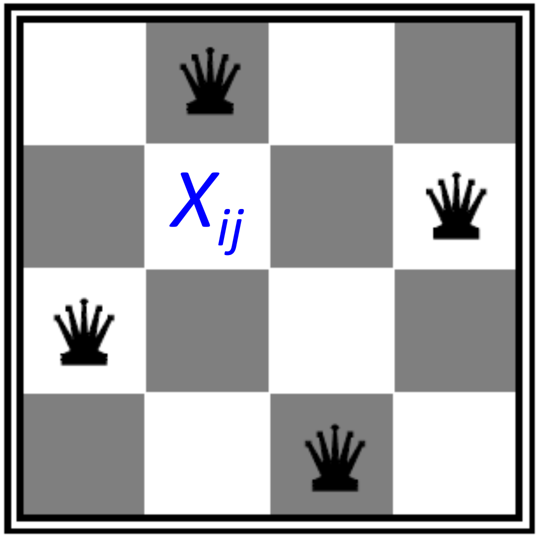
$$\Sigma_{i,j} X_{ij} = N$$

$$(X_{ij}, X_{ik}) \in \{(0, 0), (0, 1), (1, 0)\}$$

$$(X_{ij}, X_{kj}) \in \{(0, 0), (0, 1), (1, 0)\}$$

$$(X_{ij}, X_{i+k,\ j+k}) \in \{(0, 0), (0, 1), (1, 0)\}$$

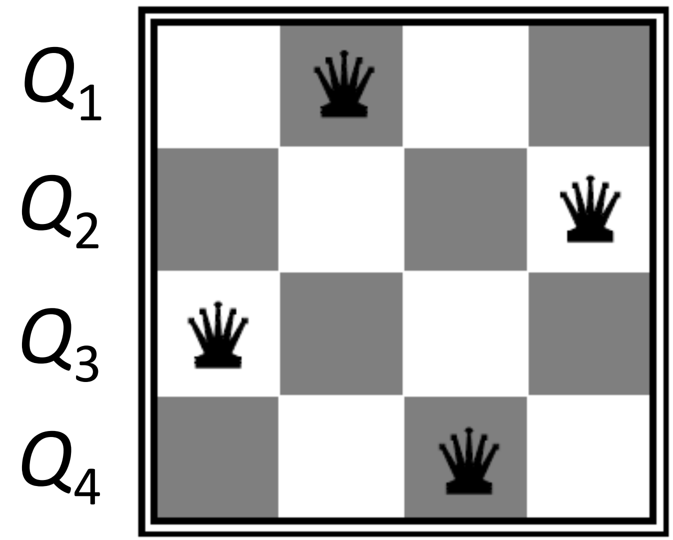
$$(X_{ij}, X_{i+k,\ j-k}) \in \{(0, 0), (0, 1), (1, 0)\}$$



N-Queens: Alternative formulation

- **Variables:** Q_i
- **Domains:** $\{1, \dots, N\}$
- **Constraints:**

  $\forall\ i, j$ non-threatening (Q_i, Q_j)



Example: Cryptarithmetic

- **Variables:** T, W, O, F, U, R

  X_1, X_2

- **Domains**: {0, 1, 2, …, 9}
- **Constraints:**

  $O + O = R + 10 * X_1$

  $W + W + X_1 = U + 10 * X_2$

  $T + T + X_2 = O + 10 * F$

  Alldiff(T, W, O, F, U, R)

  $T \neq 0, F \neq 0$

$$\begin{array}{cccc} & X_2 & X_1 & \\ & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$

Example: Sudoku

- **Variables:** X_{ij}
- **Domains:** {1, 2, …, 9}
- **Constraints:**

  Alldiff(X_{ij} in the same *unit*)

					8			4
	8	4		1	6			
			5			1		
1		3	8			9		
6		8		X_{ij}		4		3
		2			9	5		1
		7			2			
			7	8		2	6	
2			3					

Real-world CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetable problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling

- More examples of CSPs: http://www.csplib.org/

Standard search formulation (incremental)

- **States:**
  - Variables and values assigned so far
- **Initial state:**
  - The empty assignment
- **Action:**
  - Choose any unassigned variable and assign to it a value that does not violate any constraints
    - Fail if no legal assignments
- **Goal test:**
  - The current assignment is complete and satisfies all constraints

Standard search formulation (incremental)

- What is the depth of any solution (assuming ***n*** variables)?

  n (this is good)

- Given that there are ***m*** possible values for any variable, how many paths are there in the search tree?

  $n! \cdot m^n$ (this is bad)

- How can we reduce the branching factor?

Backtracking search

- In CSP's, variable assignments are **commutative**
  - For example, *[WA = red then NT = green]* is the same as *[NT = green then WA = red]*
- We only need to consider assignments to a single variable at each level (i.e., we fix the order of assignments)
  - Then there are only m^n leaves
- Depth-first search for CSPs with single-variable assignments is called **backtracking search**

Example

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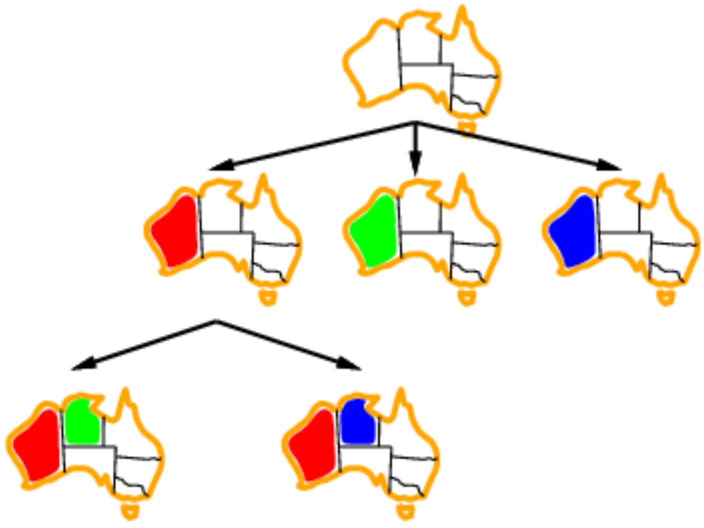

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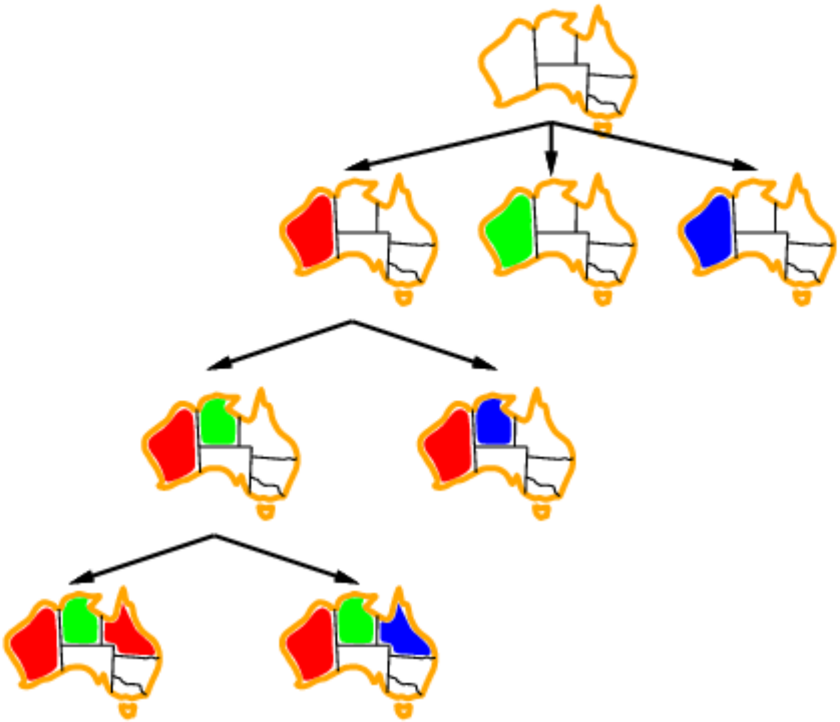

Backtracking search algorithm

function RECURSIVE-BACKTRACKING(*assignment*, *csp*)
  if *assignment* is complete **then return** *assignment*
  var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[*csp*], *assignment*, *csp*)
  for each *value* **in** ORDER-DOMAIN-VALUES(*var*, *assignment*, *csp*)
    if *value* is consistent with *assignment* given CONSTRAINTS[*csp*]
      add $\{var = value\}$ to *assignment*
      result ← RECURSIVE-BACKTRACKING(*assignment*, *csp*)
      if $result \neq failure$ **then return** *result*
      remove $\{var = value\}$ from *assignment*
  return *failure*

- Making backtracking search efficient:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

Which variable should be assigned next?

- **Most constrained variable:**
  - Choose the variable with the fewest legal values
  - A.k.a. **minimum remaining values** (MRV) heuristic

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Given a variable, in which order should its values be tried?

- Choose the **least constraining value**:
  - The value that rules out the fewest values in the remaining variables

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Which assignment for Q should we choose?

Early detection of failure

```
function RECURSIVE-BACKTRACKING(assignment, csp)
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp)
      if value is consistent with assignment given CONSTRAINTS[csp]
          add {var = value} to assignment
          result ← RECURSIVE-BACKTRACKING(assignment, csp)
          if result ≠ failure then return result
          remove {var = value} from assignment
  return failure
```

Apply *inference* to reduce the space of possible assignments and detect failure early

Early detection of failure

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Early detection of failure: Forward checking

- Keep track of remaining legal values for unassigned variables
- Terminate search when any variable has no legal values

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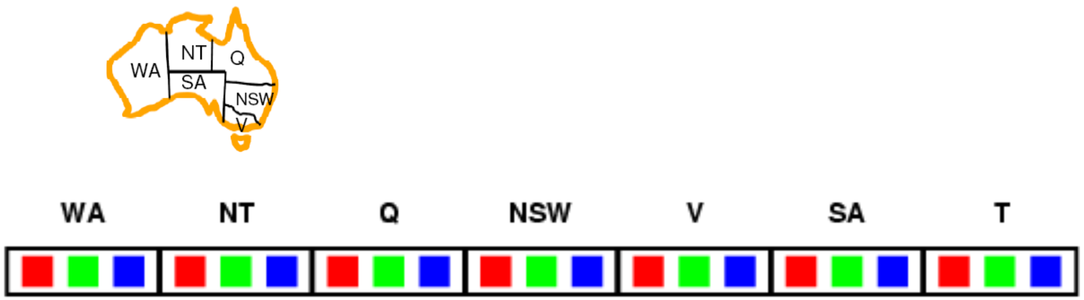

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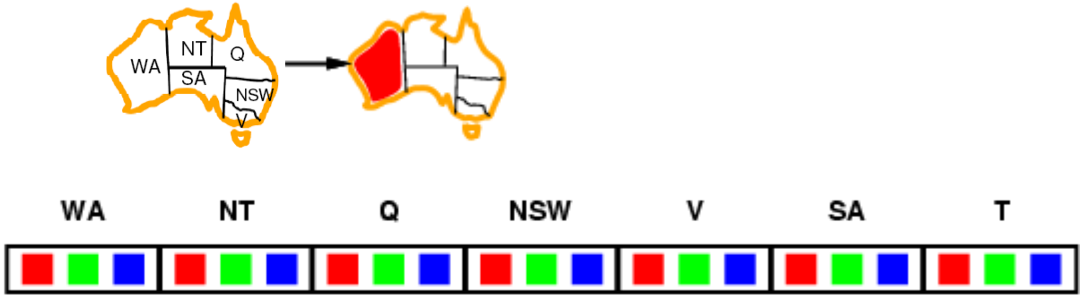

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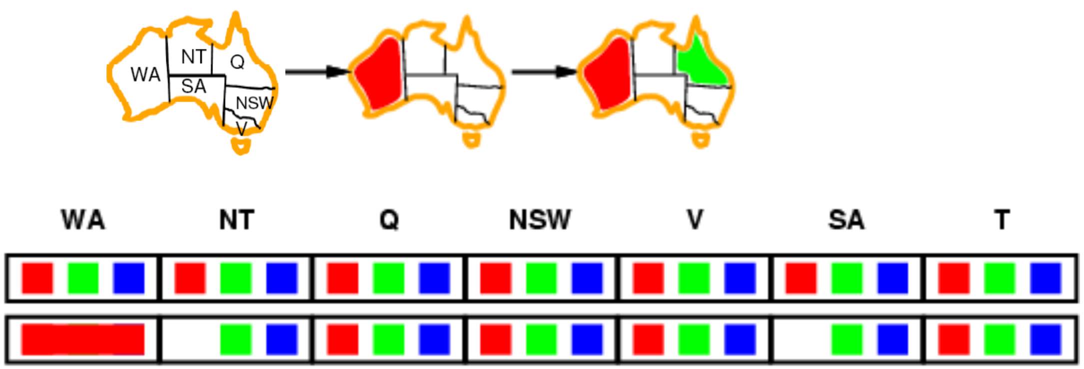

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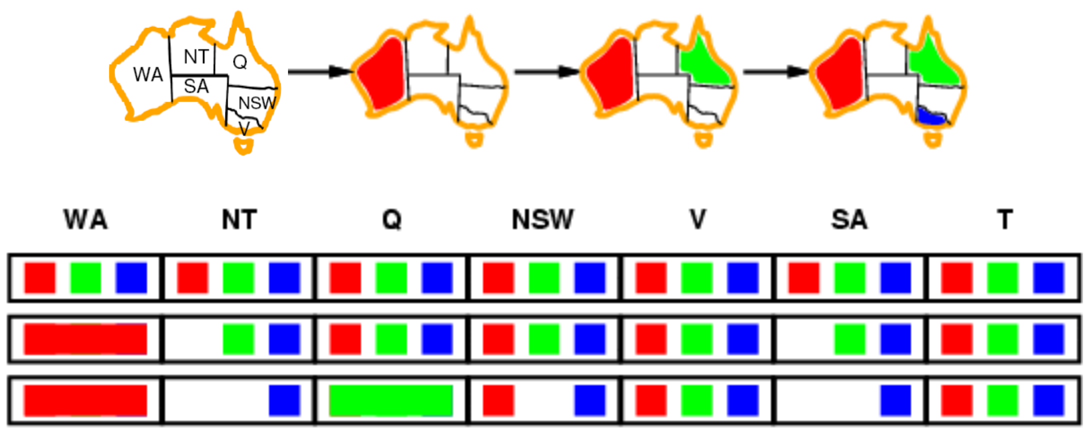
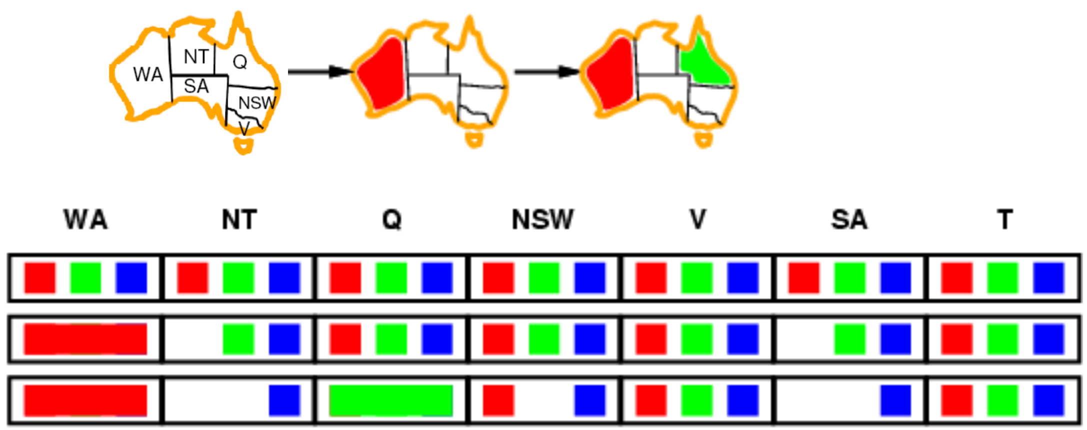


Constraint propagation

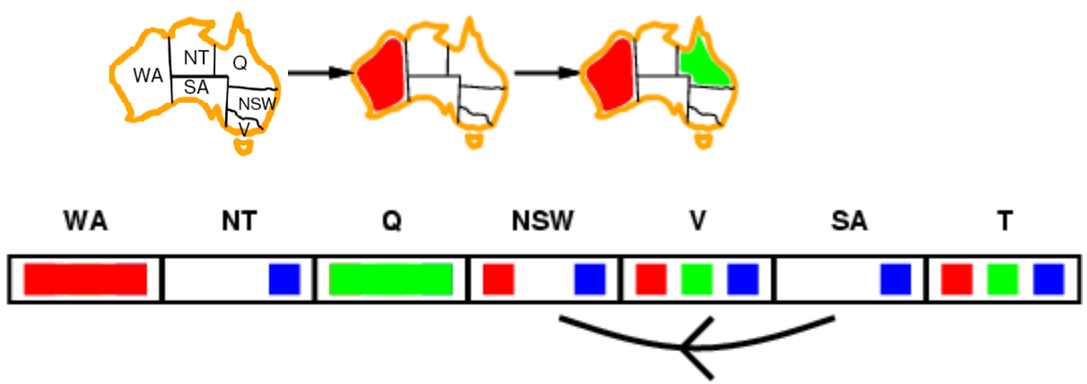
- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures



- NT and SA cannot both be blue!
- **Constraint propagation** repeatedly enforces constraints *locally*

Arc consistency

- Simplest form of propagation makes each pair of variables **consistent:**
  - *X* → *Y* is consistent iff for every value of *X* there is some allowed value of *Y*



Consistent!

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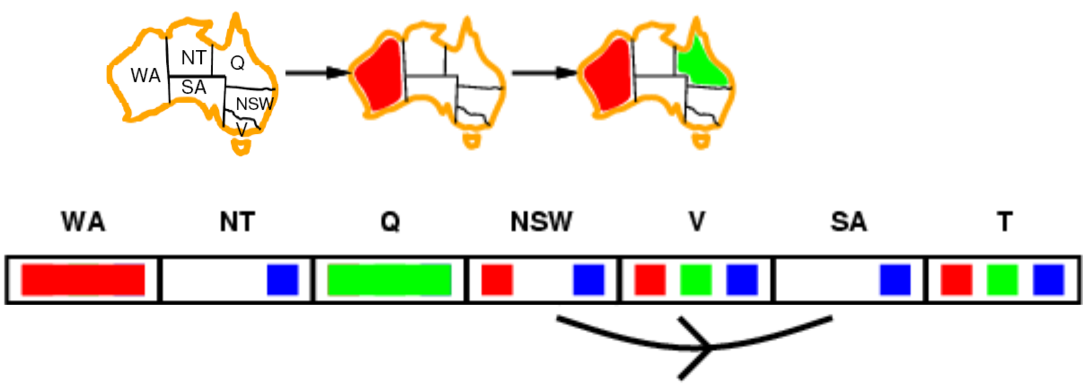
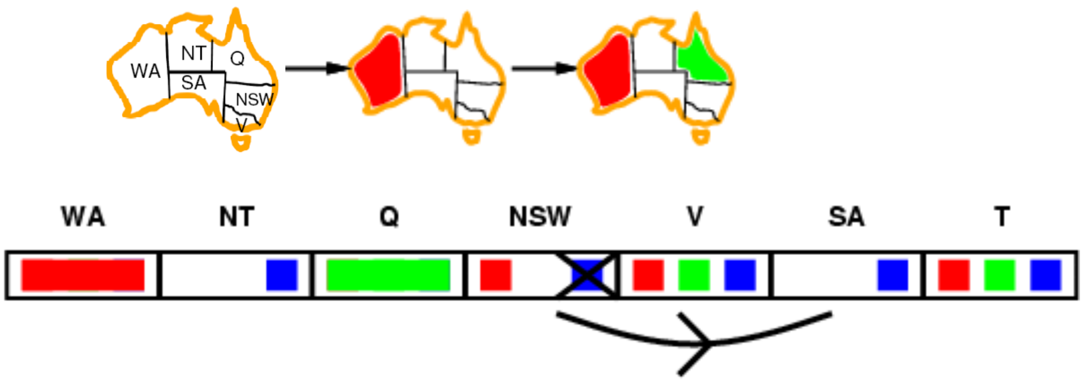


Arc consistency

- Simplest form of propagation makes each pair of variables **consistent:**
  - *X* → *Y* is consistent iff for every value of *X* there is some allowed value of *Y*
  - When checking *X* → *Y*, throw out any values of X for which there isn't an allowed value of Y



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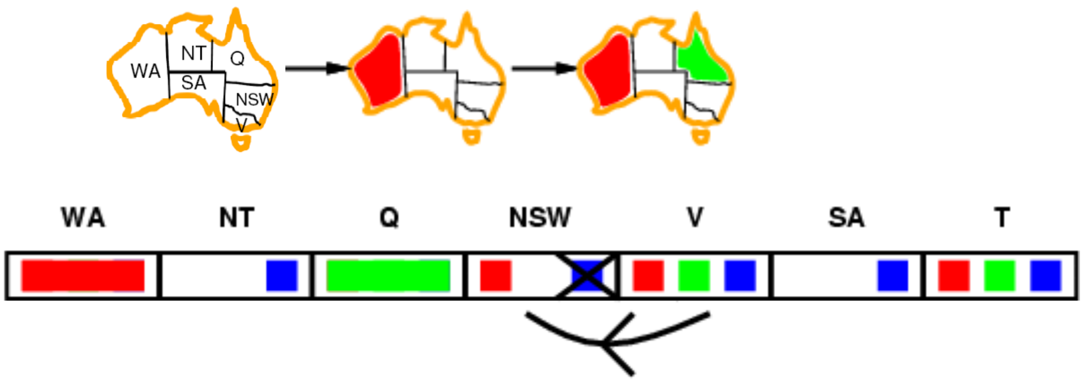

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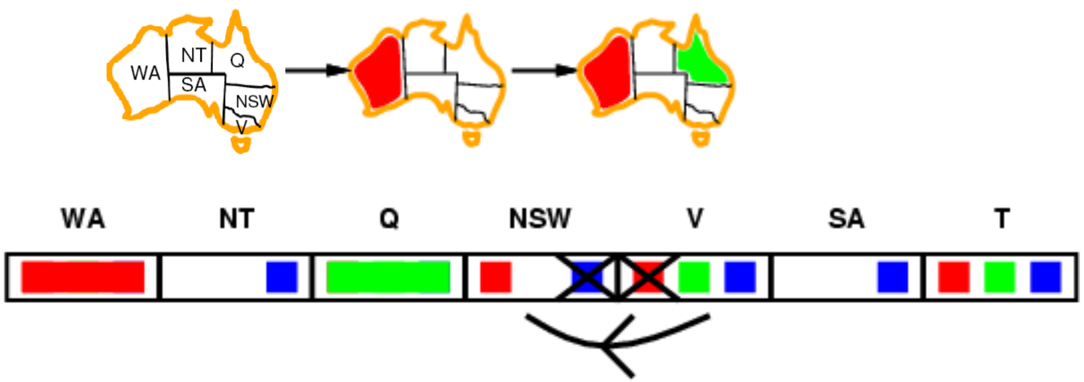

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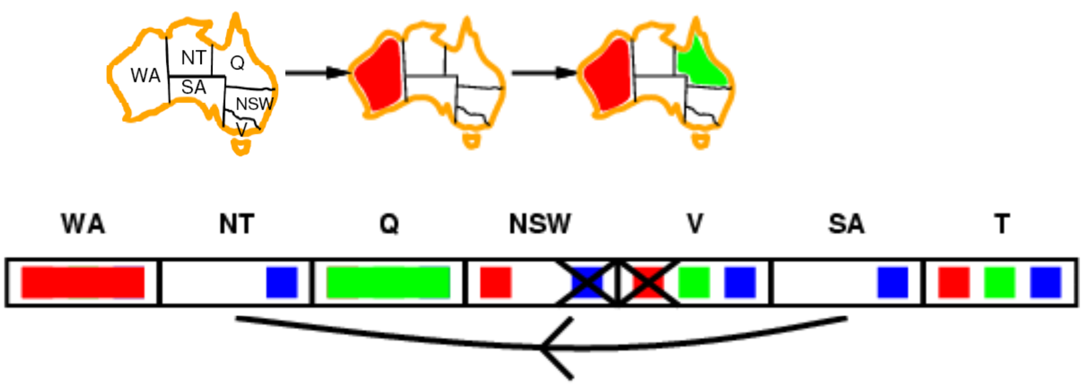
Arc consistency

- Simplest form of propagation makes each pair of variables **consistent:**
  - $X \rightarrow Y$ is consistent iff for every value of X there is some allowed value of Y
  - When checking $X \rightarrow Y$, throw out any values of X for which there isn't an allowed value of Y



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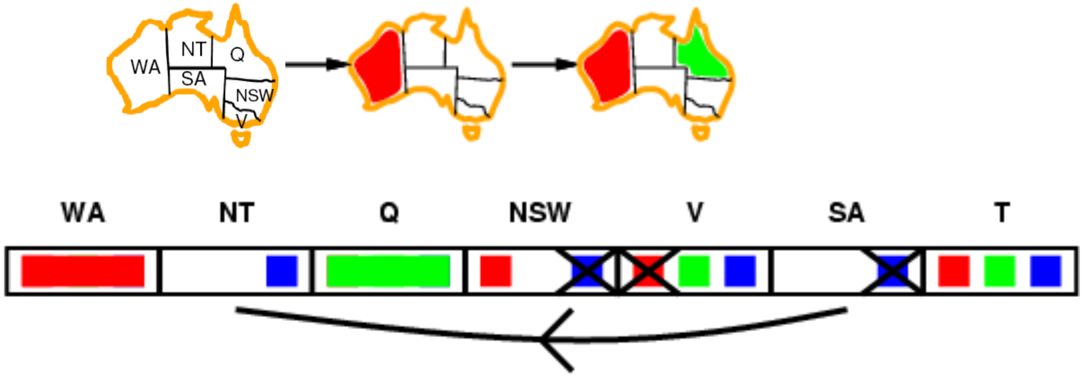


- Arc consistency detects failure earlier than forward checking
- Can be run before or after each assignment

Arc consistency algorithm AC-3

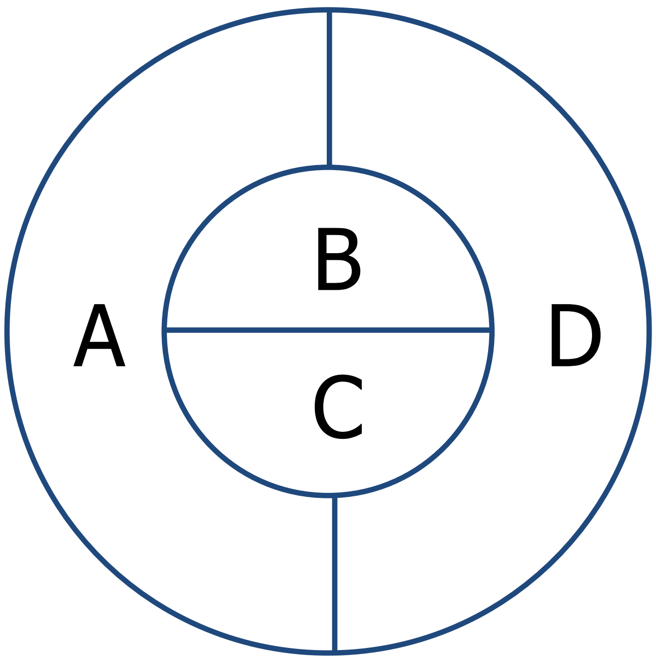
function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

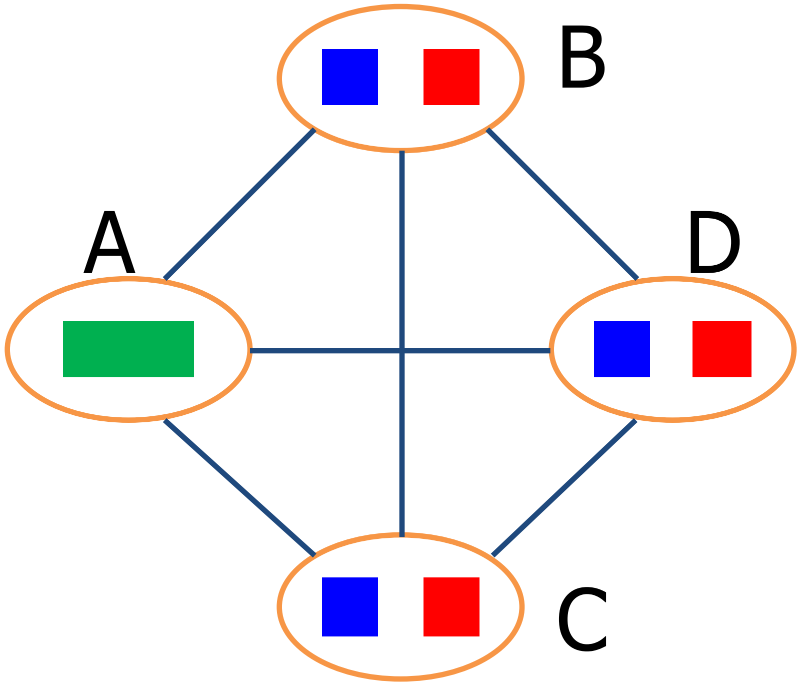
  while *queue* is not empty
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**
      for each X_k **in** NEIGHBORS[X_i] **do**
        add (X_k, X_i) to *queue*

---

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff succeeds
  removed $\leftarrow$ *false*
  for each x **in** DOMAIN[X_i]
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint $X_i \leftrightarrow X_j$
      then delete x from DOMAIN[X_i]; *removed* $\leftarrow$ *true*
  return *removed*

Does arc consistency always detect the lack of a solution?





- There exist stronger notions of consistency (path consistency, k-consistency), but we won't worry about them

Review: CSPs

- CSPs are a special kind of search problem:
  - States defined by values of a fixed set of variables
  - Goal test defined by constraints on variable values
- **Backtracking search** = DFS where successor states are generated by considering assignments to a single variable
  - **Variable ordering** and **value selection** heuristics can help significantly
  - **Forward checking** prevents assignments that guarantee later failure
  - **Constraint propagation** (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Alternatives to backtracking search
  - Local search
- Complexity of CSPs
  - NP-complete in general (exponential worst-case running time)
  - SAT and graph coloring are NP-complete and are CSPs
  - Efficient solutions possible for special cases (e.g., tree-structured CSPs)

Attribution

Slides developed by Svetlana Lazebnik based on content from Stuart Russell and Peter Norvig, Artificial Intelligence: A Modern Approach, 3rd edition