CSCI 340: Computational Models

Non-Context-Free Languages

Chapter 16

Department of Computer Science

Self-Embeddedness

Theorem

*Let **G** be a CFG in Chomsky Normal Form. Let us call the production of the form:*

$$\textit{Nonterminal} \quad \longrightarrow \quad \textit{Nonterminal Nonterminal}$$

live *and the productions of the form*

$$\textit{Nonterminal} \quad \longrightarrow \quad \textit{terminal}$$

dead. *If we restrict to using live productions at most once each, we can generate only finitely many words.*

Self-Embeddedness

- Every time we apply a **live** production, we increase the number of nonterminals by one
- Every time we apply a **dead** production, we decrease the number of nonterminals by one
- We will always apply one more **dead** production than **live** productions.
- Show the *self-embeddedness* of any word generated by S

Example

$$S \longrightarrow AZ$$
$$Z \longrightarrow BB$$
$$B \longrightarrow ZA$$
$$A \longrightarrow a$$
$$B \longrightarrow b$$

Self-Embeddedness

Note

When we expand the productions of a grammar in CNF, we will always produce a **binary tree** as our *derivation tree*.

Because of this property, we can theoretically construct a *complete* binary tree

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Theorem

If G is a CFG in CNF that has p live productions and q dead productions, and if w is a word generated by G that has more than 2^p letters in it, then somewhere in every derivation tree for w there is an example of some nonterminal (call it Z) being used twice where the second Z is descended from the first Z.

Self-Embeddedness

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Theorem

If $\boldsymbol{G}$ is a CFG in CNF that has $\boldsymbol{p}$ live productions and $\boldsymbol{q}$ dead productions, and if $\boldsymbol{w}$ is a word generated by $\boldsymbol{G}$ that has more than 2^p letters in it, then somewhere in every derivation tree for $\boldsymbol{w}$ there is an example of some nonterminal (call it $\boldsymbol{Z}$) being used twice where the second $\boldsymbol{Z}$ is descended from the first $\boldsymbol{Z}$.

The **live** productions indicate the maximum *depth* of the tree

Self-Embeddedness

Definition

In a given derivation of a word in a given CFG, a nonterminal is said to be **self-embedded** if it ever occurs as a tree descendent of itself

Example

CFG for NONNULLPALINDROME — derivation for *aabaa*

$$S \rightarrow AX \qquad S \rightarrow b$$
$$X \rightarrow SA \qquad S \rightarrow AA$$
$$S \rightarrow BY \qquad S \rightarrow BB$$
$$Y \rightarrow SB \qquad A \rightarrow a$$
$$S \rightarrow a \qquad B \rightarrow b$$

Self Embeddedness

Definition

Let us introduce the notation $\overset{*}{\Rightarrow}$ to stand for the phrase "can eventually produce". It is used in the following context:

Suppose in a certain CFG the working string S_1 can produce the working string S_2, which in turn can produce $S_3 \ \ldots \ S_n$

We can then write:

$$S_1 \overset{*}{\Rightarrow} S_n$$

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For NONNULLPALINDROME, we can state the following:

$$X \stackrel{*}{\Rightarrow} a^n X a^n$$

Non-Context-Free Languages

- It turns out that not all languages are context-free.
- The simplest example of a non-context-free language is

$$\{\mathbf{a}^n\mathbf{b}^n\mathbf{c}^n \mid n \geq 0\}.$$

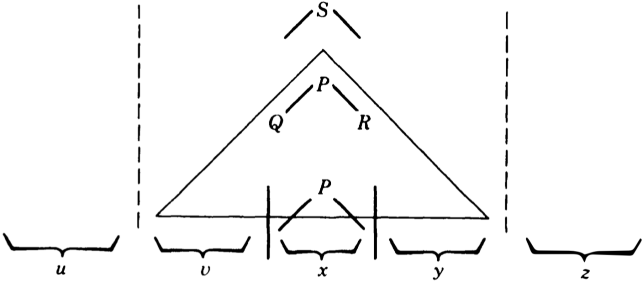
- To process this would require two stacks.

The Pumping Lemma

Theorem (The Pumping Lemma for Context-Free Grammars)

If L is a context-free language then there exists an integer p such that if any string $s \in L$ has length at least p, then s may be divided into five substrings $s = uvxyz$ such that

- $|vy| > 0$,
- $|vxy| \leq p$,
- $uv^ixy^iz \in L$ *for all* $i \geq 0$.



The Pumping Lemma Parts

- u — the substring of all the letters of w generated to the "left" of the derivation we care about
- v — the substring of all the letters of w descended from the root of the derivation we care about but to the left of the **self-embedded** state
- x — the substring of all the letters of w descended from the **self-embedded** state
- y — the substring of all the letters of w descended from the right of the **self-embedded** state to the end of the derivation we care about
- z — the substring of all the letters of w generated to the "right" of the derivation we care about

The Proof

- The proof is somewhat similar to the proof of the Pumping Lemma for Regular Languages except that it is based on a grammar rather than a machine.
- But first, an example…

An Example

Example

- Let $L = \{\mathbf{a}^n\mathbf{b}^n\mathbf{c}^n \mid n \geq 0\}$.
- We will use the Pumping Lemma to show that L is not context-free.

An Example

Proof.

- Suppose L is context-free.
- Then let p be the "pumping length" of L (for CFLs).
- Let $s = \mathbf{a}^p\mathbf{b}^p\mathbf{c}^p$.
- Then $s = uvxyz$ such that $|vy| > 0$, $|vxy| \leq p$, and $uv^ixy^iz \in L$.
- We will show that this is not possible.

$\square$

An Example

Proof.

- *vxy* is the "middle part" of *uvxyz* and it has length at most *p*.
- Therefore, it consists of

  Case 1: All **a**'s,
  Case 2: Some **a**'s followed by some **b**'s,
  Case 3: All **b**'s,
  Case 4: Some **b**'s followed by some **c**'s, or
  Case 5: All **c**'s.

$\square$

An Example

Proof.

- It is enough to consider the first two cases.
- The other three cases are similar.
- Case 1: Suppose vxy consists of all $\mathbf{a}$'s.
  - Then $v = \mathbf{a}^k$ and $y = \mathbf{a}^m$ for some k, m, not both 0.
  - So $uv^2xy^2z = \mathbf{a}^{p+k+m}\mathbf{b}^p\mathbf{c}^p$, which is not in L.
  - This is a contradiction.

$\square$

An Example

Proof.

- Case 2: vxy consists of some **a**'s followed by some **b**'s.
  - There are three possibilities:
    - v is all **a**'s and y is all **b**'s,
    - v is all **a**'s and y is some **a**'s followed by some **b**'s,
    - v is some **a**'s followed by some **b**'s and y is all **b**'s.

□

An Example

Proof.

- Case 2, continued…
  - It doesn't really matter which is the case because both v and y get pumped up.
  - Let k be the number of **a**'s and m be the number of **b**'s altogether in vy, m and k are not both 0 (but possibly $m = k$).
  - So uv^2xy^2z will contain $p + k$ **a**'s and $p + m$ **b**'s, but only p **c**'s.
  - So $uv^2xy^2z \notin L$.
  - This is a contradiction.
- Cases 3, 4, and 5 are similar.
- Therefore, L is not context-free.

$\square$

The Idea Behind the Proof

- If a CFL contains a string w with a sufficiently long derivation

  $$S \overset{*}{\Rightarrow} w,$$

  then some variable A must appear more than once in the derivation.
- That is, we must have

  $$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvxyz,$$

  for some strings u, v, x, y, and z.

The Idea Behind the Proof

- Thus, $A \overset{*}{\Rightarrow} vAy$ and $A \overset{*}{\Rightarrow} x$.
- We may repeat the derivation

$$A \overset{*}{\Rightarrow} vAy$$

  as many times as we like (including zero times), producing strings uv^nxy^nz, for any $n \geq 0$.

The Proof

Proof.

- Let b be the largest number of symbols on the right-hand side of any grammar rule. (Assume $b \geq 2$.)
- Let h be the height of the derivation tree of a string s.
- Then s can contain at most b^h symbols.
- Equivalently, if s contains more than b^h symbols, then the height of the derivation tree of s must be more h.

□

The Proof

Proof.

- Now $|V|$ is the number of variables in the grammar of L.
- So if a string in L has a length greater than $b^{|V|+1}$, then the height of its derivation tree must be more than $|V| + 1$.
- So let $p = b^{|V|+1}$ and suppose that a string $s \in L$ has length at least p.

$\square$

The Proof

Proof.

- Consider the longest path through the derivation tree of s.
- It has length at least $|V| + 1$.
- That path has $|V| + 2$ nodes on it, counting the root node S and the leaf node, which is a terminal.

$\square$

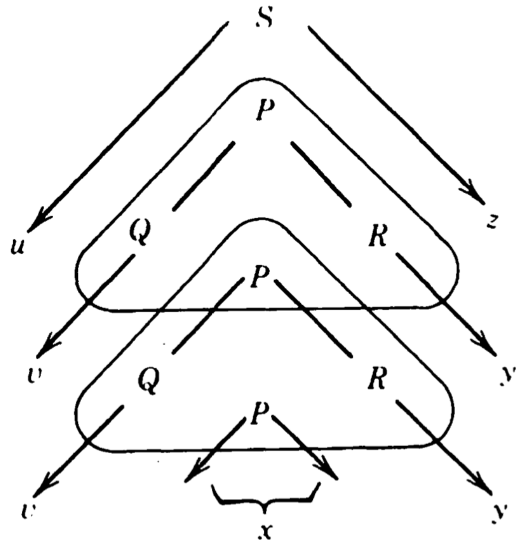
The Proof

Proof.

- Thus, $|V| + 1$ of the nodes are variables.
- So one of them must be repeated.
- As we follow the longest path back from leaf to root, let A be the first variable that repeats.
- Now consider these two occurrences of A along the longest path.

$\square$

The Proof



The Proof

Proof.

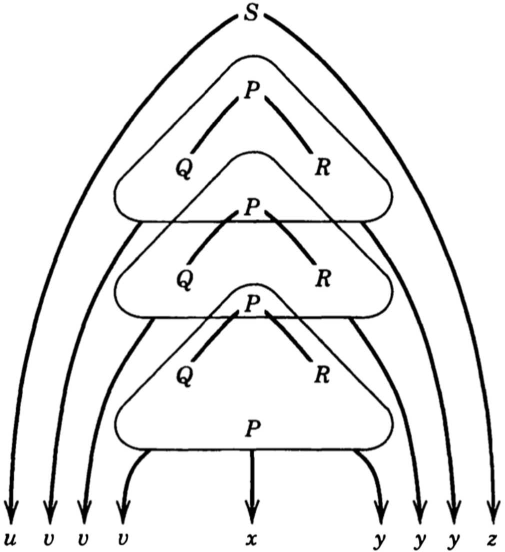
- The "middle part" of this tree, the part that produces

$$A \stackrel{*}{\Rightarrow} vAy,$$

may be repeated as many times as desired.

□

The Proof



The Proof

Proof.

- Therefore, the strings uv^2xy^2z, uv^3xy^3z, etc. can also be derived.
- So can the string uxz.
- Furthermore, we may assume that this was the shortest derivation of s.
- It follows that v and y cannot both be empty strings.
- If they were, then the middle part of the derivation would be

$$A \stackrel{*}{\Rightarrow} A,$$

  which could be eliminated.
- Thus, $|vy| > 0$.

$\square$

The Proof

Proof, conclusion.

- Finally, we must show that $|vxy| \leq p$.
- The subtree rooted at the second-to-last A has height at most $|V| + 1$.
- So the string vxy has at most $b^{|V|+1} = p$ symbols.

$\square$

An Example

Example

- Let $\Sigma = \{\mathbf{a}, \mathbf{b}\}$.
- Show that the language

$$\{ww \mid w \in \Sigma^*\}$$

  is not context-free.
- Use $s = \mathbf{a}^p\mathbf{b}^p\mathbf{a}^p\mathbf{b}^p$.

Homework 9a

1. Consider the grammar for the language $L = \{ a^n b^n \}$
   1. (5pts) Chomsky-ize this grammar
   2. (5pts) Find all derivation trees that **do not** have self-embedded non-terminals
2. (5pts) Why does the pumping lemma argument **not** show the language PALINDROME is not context free? Show how v and y can be found such that $w = uv^n xy^n z$ are also in PALINDROME no matter what w is.
3. (5pts) How would you go about proving the following theorem? If L is a language over the one-letter alphabet $\Sigma = \{ a \}$ and L can be shown to be non-regular using the pumping lemma for regular languages, then L can be shown to be non-context-free using the pumping lemma for context-free languages.