CSCI 340: Computational Models

Regular Languages

Chapter 9

Department of Computer Science

Regular Languages

If we can define a language by RE, then it's a *regular language*

Theorem

If L_1 and L_2 are regular languages, then $L_1 + L_2$ (union) , $L_1 L_2$ (concatenation), and $L_1{}^*$ (closure) are also regular languages.

Proof by Regular Expression.

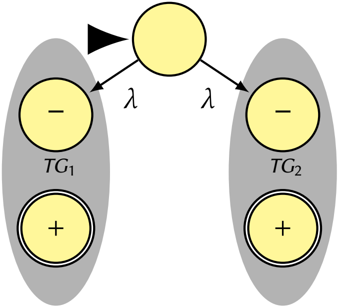
1. There exists REs $\mathbf{r_1}$ and $\mathbf{r_2}$ that define the regular languages L_1 and L_2
2. There exists an RE $(\mathbf{r_1} + \mathbf{r_2})$ that defines the language $L_1 + L_2$
3. There exists an RE $\mathbf{r_1 r_2}$ that defines the language $L_1 L_2$
4. There exists an RE $\mathbf{r_1}^*$ that defines the language $L_1{}^*$
5. All three of these sets of words are definable by RE □

The set of regular languages is *closed* under union, concatenation, and Kleene closure.

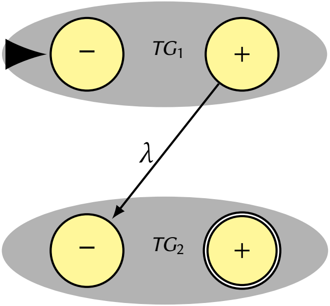
Proof by Machines

1. Let us assume TG_1 and TG_2 exist that define languages L_1 and L_2 where each TG has a unique start and final state

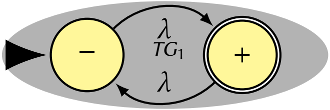
2. $L_1 + L_2$ can be described by:



3. $L_1 L_2$ can be described by:



4. $L_1{}^*$ can be described by:



□

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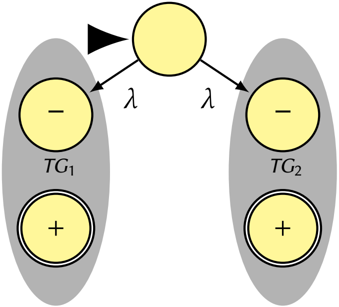

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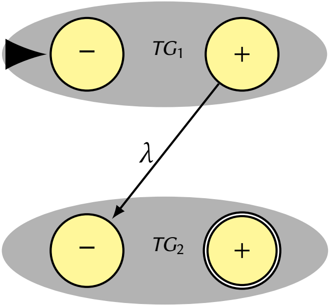

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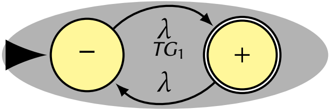

Small problem for $L_1{}^*$ when the start has incoming edges. We must replicate the start state. We could convert to FA-λ then to FA. □

Example

$$\Sigma = \{a \quad b\}$$

L_1 = all words of 2+ letters that begin and end with the same letter

L_2 = all words that contain the substring *aba*

$$\mathbf{r_1} = \mathbf{a}(\mathbf{a} + \mathbf{b})^*\mathbf{a} + \mathbf{b}(\mathbf{a} + \mathbf{b})^*\mathbf{b}$$

$$\mathbf{r_2} = (\mathbf{a} + \mathbf{b})^*\mathbf{aba}(\mathbf{a} + \mathbf{b})^*$$

$$\mathbf{r_1} + \mathbf{r_2} =$$

$$\mathbf{r_1}\mathbf{r_2} =$$

$$\mathbf{r_1}^* =$$

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Show the TGs that accept L_1 and L_2

Show $TG_1 + TG_2$, $TG_1 TG_2$, and $TG_1{}^*$

Complements and Intersections

Definition

If L is a language over alphabet Σ, we define its **complement**, L' to be the language of all strings of letters from Σ that are *not* words in L.

Example

If L is the language over the alphabet $\Sigma = \{a \quad b\}$ of all words that have a double a in them, then L' is the language of all words that do not have a double a.

We must specify the alphabet Σ or else the complement of L might contain *cat*, *dog*, . . . (because they are definitely not strings in L).

$$(L')' = L$$

for obvious reasons (theorem in set theory)

Complements and Regular Languages

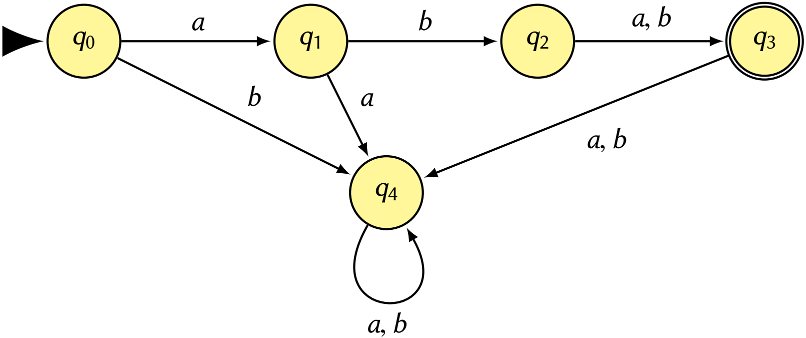
Theorem

If L is a regular language, then L' is also a regular language. In other words, the set of regular languages is closed under complementation.

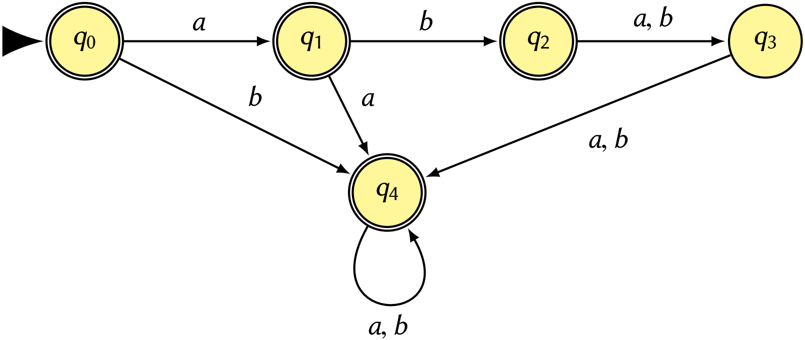
Proof.

- If L is a regular language, we know from Kleene's theorem that there is some FA that accepts L.
- The states of FA are each either final or non-final
- Let us reverse the final status of each state (e.g. final $\rightarrow$ non-final, non-final $\rightarrow$ final)
- This new machine accepts all input strings the original FA rejected (L'). Likewise, the new machine rejects all input strings the original FA accepted (L).
- This new FA can be converted to an RE via Kleene's theorem □

Complements of Regular Languages Example



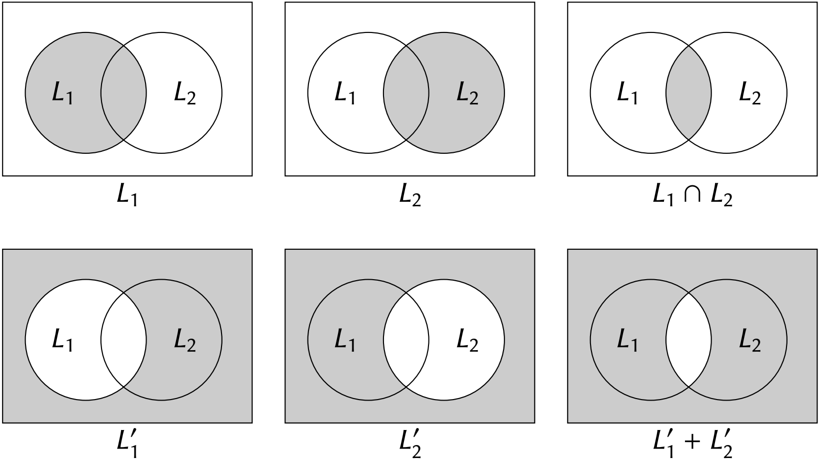
Complements of Regular Languages Example



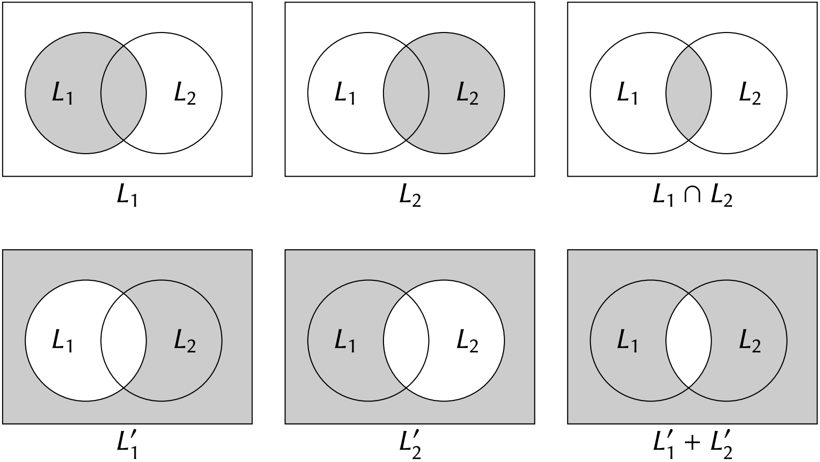
Language Intersection

Theorem

If L_1 and L_2 are regular languages, than $L_1 \cap L_2$ is also a regular language. e.g. the set of regular languages is closed under intersection.



Language Intersection



From the above, it is obvious how $(L_1' + L_2')' = L_1 \cap L_2$

Algorithm for finding RE accepting $L_1 + L_2$

Algorithm

1. Define $\mathbf{r_1}$ and $\mathbf{r_2}$ which represent L_1 and L_2
2. Convert $\mathbf{r_1}$ and $\mathbf{r_2}$ to FA_1 and FA_2
3. Invert the states of FA_1 and FA_2 resulting in FA_1' and FA_2'
4. Merge FA_1' and FA_2' into TG', then convert TG' into FA_3'
5. Invert the states of FA_3', resulting in FA_3 (which accepts $L_1 \cap L_2$)

Proof.

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4. Merge FA_1' and FA_2' into TG', then convert TG' into FA_3'
5. Invert the states of FA_3', resulting in FA_3 (which accepts $L_1 \cap L_2$)

Proof.

1. For a regular language, there exists a RE
2. Given an RE, there exists an FA (Kleene's theorem)
3. We can complement an FA by swapping its states
4. We can describe $L_1' + L_2'$ by merging two TGs
5. We can convert a TG to an RE □

Example

L_1 = all strings with a double a

L_2 = all strings with an even number of a's

Example

$$L_1 = \text{all strings with a double} a$$
$$L_2 = \text{all strings with an even number of } a\text{'s}$$

We can define L_1 and L_2 by the following REs:

$$\mathbf{r_1} = (\mathbf{a} + \mathbf{b})^* \mathbf{aa} (\mathbf{a} + \mathbf{b})^*$$
$$\mathbf{r_2} = \mathbf{b}^* (\mathbf{ab}^* \mathbf{ab}^*)^*$$

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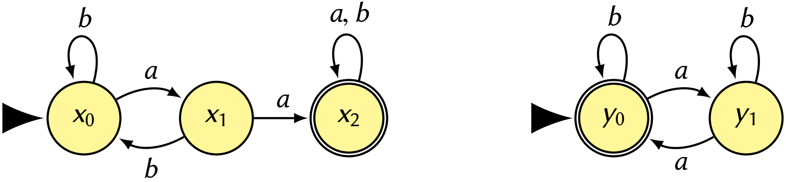
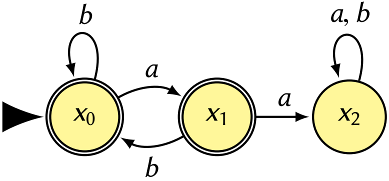
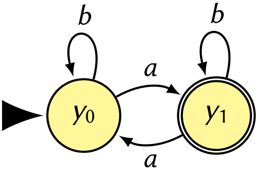
Or the following FAs:



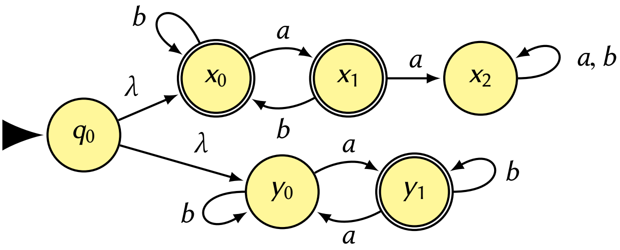
Example

Swapping the states:



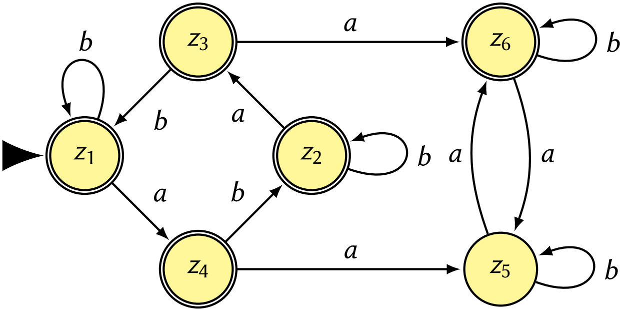


Merging (Creating the TG):



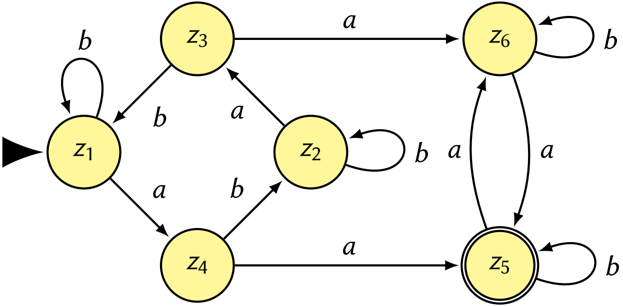
Example

After converting the TG to FA:



Example

After swapping all of the states:



And converting the FA to RE with the bypass algorithm:

$$(\mathbf{a} + \mathbf{abb}^*\mathbf{ab})^*\mathbf{a}(\mathbf{a} + \mathbf{bb}^*\mathbf{aab}^*\mathbf{a})(\mathbf{a} + \mathbf{ab}^*\mathbf{a})^*$$

A Better Way...

- Remember creating a machine that accepts $FA_1 + FA_2$ where FA_1 has x-states, FA_2 has y-states, and our new machine has z-states
- We identify all final z-states by x-or-y states being accepted upon the construction of our new machine
- Let's change the designation for $FA_1 \cap FA_2$ to:
  All final z-states by x-and-y states being accepted upon the construction of our new machine
- Now the new FA accepts only strings that reach simultaneously on both machines

TL;DR – change the rules of determining a final state of two FAs to be the intersection ($\cap$) rather than union ($+$)

One Final Example

Our two languages will be:

$$
\begin{aligned}
L_1 &= \text{all words that begin with an} a \\
L_2 &= \text{all words than end with an} a \\
\mathbf{r_1} &= \mathbf{a}(\mathbf{a} + \mathbf{b})^* \\
\mathbf{r_2} &= (\mathbf{a} + \mathbf{b})^*\mathbf{a}
\end{aligned}
$$

An obvious solution is:

$$\mathbf{a}(\mathbf{a} + \mathbf{b})^*\mathbf{a} + \mathbf{a}$$

But now we need to prove it...

Homework 6a

For each of the following pars of regular languages, find a RE and FA that define $L_1 \cap L_2$

1.	$(\mathbf{a} + \mathbf{b})^*\mathbf{a}$	$\mathbf{b}(\mathbf{a} + \mathbf{b})^*$
2.	Even-length strings	$(\mathbf{b} + \mathbf{ab})^*(\mathbf{a} + \lambda)$
3.	Odd-length strings	$\mathbf{a}(\mathbf{a} + \mathbf{b})^*$
4.	Even-length strings	Strings with an even number of a's