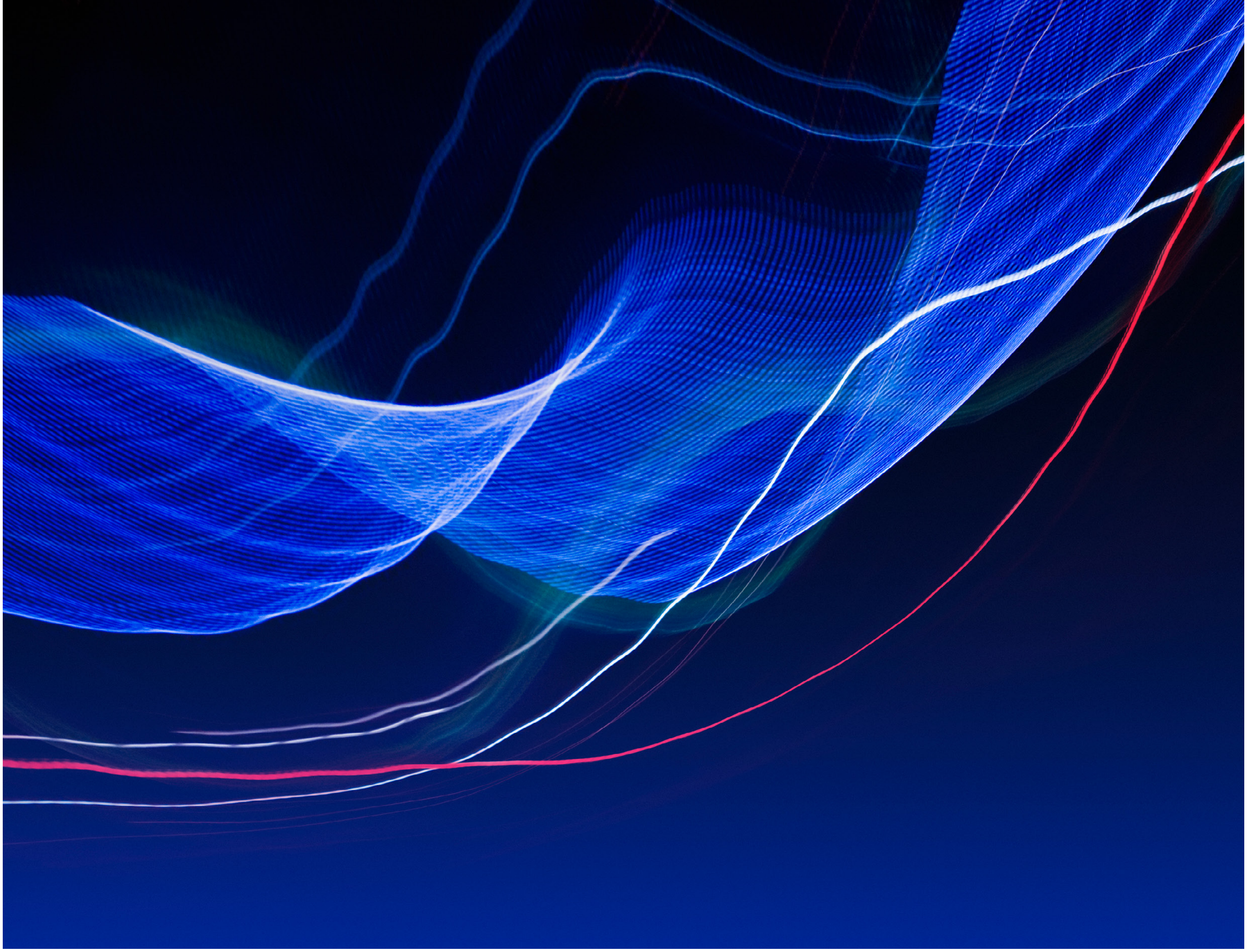
ESSVI IMPLIED VOLATILITY SURFACE

Amir Akhundzadeh, Timothy Huber, Matthew Hyatt, Eric Schneider



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1 Introduction

The Stochastic Volatility Inspired (SVI) parametrization for total variance was introduced by Gatheral (see [2]). It is a five-parameter formula that fits market data remarkably well for a given maturity. Let k denote the log forward moneyness (i.e., $k = \ln(K/F)$ where K is the strike and F the forward price of the underlying), and let w denote the total variance (i.e., $w = t\sigma^2$ where t is the maturity and σ is the implied volatility). The SVI parametrization is

$$w(k) = a + b\left[\rho(k-m) + \sqrt{(k-m)^2 + c^2}\right]$$

where a, b, ρ, m, and c are the parameters. We note that b is assumed nonnegative, c is assumed positive, and $|\rho| \leq 1$ as this parameter represents the correlation between processes of the underlying and the volatility. Other reasons for the popularity of this model include the fact that it has linear asymptotic growth, as expected from the work of Lee in [6]. We note that Lee's result is model-independent. Furthermore, in [3] it is shown that the Heston implied volatility model converges to the SVI parametrization in the long maturity limit. Although one can interpolate SVI slices to create a volatility surface, this surface is often unsatisfactory due to the presence of static arbitrage (see [4] for definition). For example, the presence of calendar spread arbitrage can easily be seen in the left panel of Figure 1 since $\partial_t w \geq 0$ is most definitely violated.

To address these issues, Gatheral and Jacquier introduced the Surface Stochastic Volatility Inspired (SSVI) parametrization [4]. Letting θ_t denote the forward at-the-money (ATM) implied total variance, the surface is parametrized by

$$w(k, \theta_t) = \frac{\theta_t}{2}\left[1 + \rho\varphi(\theta_t)k + \sqrt{(\varphi(\theta_t)k + \rho)^2 + 1 - \rho^2}\right],$$

where φ is a smooth, positive-valued function such that $\lim_{t\to 0} \theta_t \varphi(\theta_t)$ exists. At a fixed maturity, this amounts to a three-parameter specialization of the SVI parametrization, with the caveat that ρ is constant across all maturities. The trade-off here is that the parametric form is now more rigid, having fewer parameters, but in return Gatheral and Jacquier provide tractable conditions for the preclusion of calendar spread arbitrage and butterfly arbitrage.

In order to increase flexibility of the parametric form and thus improve the fit to market data, Hendriks and Martini introduced the extended Stochastic Volatility Inspired (eSSVI) parametrization [5]. They allow the parameter ρ to vary

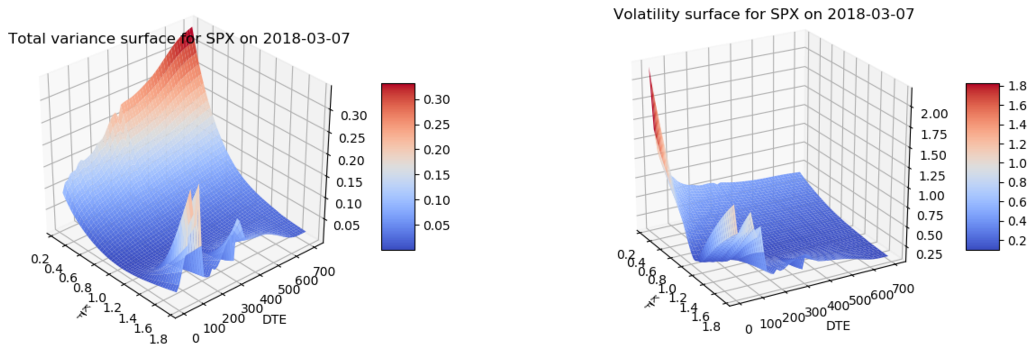

Figure 1: SVI implied volatility and total variance surface for SPX on March 7, 2018.

with maturity and prove extended versions of the calendar spread and butterfly arbitrage conditions, which are still tractable. Following [1], we implement the eSSVI parametrization in the form

$$w(k) = \frac{1}{2}\left[\theta + \rho\psi k + \sqrt{(\psi k + \rho\theta)^2 + (1-\rho^2)\theta^2}\right], \tag{1}$$

where $\theta = \theta^* - \rho\psi k^*$ and the parameters ρ, ψ, θ^*, and k^* may depend on maturity (we omit a t subscript for notational convenience). The parameters k^* and θ^* are taken directly from market data: k^* is the log forward moneyness that is closest to ATM, and θ^* is the corresponding total implied variance. Here it is assumed that $\psi > 0$ and $|\rho| < 1$. We note that the remaining parameters ρ and ψ are *not* determined by straightforward regression. In order to meet a sufficient condition to preclude butterfly arbitrage, we must ensure that

$$\psi \leq \frac{4}{1+|\rho|}, \tag{2}$$

and

$$\psi \leq \frac{-2\rho k^*}{1+|\rho|} + \sqrt{\frac{(2\rho k^*)^2}{(1+|\rho|)^2} + \frac{4\theta^*}{1+|\rho|}}. \tag{3}$$

To ensure there is no calendar spread arbitrage, we must consider slices taken at two separate maturities $t_1 < t_2$. Let $(\theta_1, \rho_1, \psi_1)$ and $(\theta_2, \rho_2, \psi_2)$ be the parameters corresponding to maturities t_1 and t_2 respectively. There is no calendar spread arbitrage between these slices if and only if the following three inequalities are satisfied

$$\theta_1 \leq \theta_2, \tag{4}$$

$$\psi_1 \leq \psi_2, \tag{5}$$

$$|\rho_2\psi_2 - \rho_1\psi_1| \leq \psi_2 - \psi_1. \tag{6}$$

2 Algorithm

To determine the parameters ρ and ψ at each maturity, we leverage the algorithm found in [1] and describe our implementation of it. At each maturity, assume we have a reliable set of market data points $\{(w_j, k_j, v_j)\}$, which are the total implied variance, log-moneyness, and vega respectively. Implied volatility for each security is computed using the Cox-Ross-Rubinstein Binomial Option Pricing Model for American style options, and the Black-Scholes model is used for European style options. For more details on the data preparation, see Appendix A. Let $t_1 < t_2 < \cdots < t_n$ denote the maturities where data exists and let ρ_i and ψ_i be the values of the parameters at maturity t_i which are to be determined. Also, as above let $\theta_i = \theta^* - \rho_i \psi_i k^*$ where k^* is the data point closest to ATM and θ^* is the corresponding value of total implied variance.

We choose a vega-weighted objective function of the form

$$f(\rho, \psi) = \sum_j v_j \left[w_j - w(k_j, \rho, \psi)\right]^2 \tag{7}$$

where

$$w(k_j, \rho, \psi) = \frac{1}{2}\left[\theta + \rho\psi k_j + \sqrt{(\psi k_j + \rho\theta)^2 + (1-\rho^2)\theta^2}\right]$$

as in the expression in (1). At each maturity, our goal is to find the minimum value of the objective function f subject to the constraints (2)-(6). We solve this problem sequentially starting from the shortest maturity and ending with the longest. At each maturity we proceed as follows:

1. Sample ρ in the interval $(-1, 1)$, say 20 evenly spaced values.
2. For each value of ρ, determine bounds $\psi_{\min}$ and $\psi_{\max}$ such that the inequalities (2)-(6) are satisfied for all values of ψ in the interval $[\psi_{\min}, \psi_{\max}]$. For (4)-(6), the values ρ_1, ψ_1, and θ_1 represent the parameters at the previous slice, and thus their values are fixed. ρ_2, ψ_2, and θ_2 represent the parameters we are currently trying to determine. Note that for the first maturity t_1, we need only to satisfy (2) and (3) with $\psi > 0$.
3. For each fixed value of ρ, solve the 1-dimensional optimization problem of minimizing f such that $\psi_{\min} \le \psi \le \psi_{\max}$.
4. Store the values of ρ and ψ that yield the smallest value of the objective function f.
5. Repeat steps 1-4 with a smaller interval centered at the value of ρ found in the previous step. The bounds of this interval can be the neighboring values of ρ from the previous sampling.

A few items for consideration regarding the above algorithm: first, we find it sufficient to run through the above loop a pre-determined number of times, although one could choose some other stopping criteria. Secondly, it is possible that the interval $[\psi_{\min}, \psi_{\max}]$ is empty, in which case we skip the calibration at this maturity. If more than 30% of maturities are skipped, then no surface will be calculated.

3 Calibration Results

We use a weighted root mean square error (WRMSE) of volatility to compare the fitting error of the eSSVI surface to other models. Since our objective function is vega-weighted, we include those weights here as well,

$$\text{WRMSE} = \sqrt{\frac{\sum_{i,j} v_j \left[\sqrt{\frac{w_j}{t_i}} - \sqrt{\frac{w(k_j)}{t_i}}\right]^2}{\sum_{i,j} v_j}}$$

where i runs through all maturities and j runs through all data points at each maturity. Note that the quantity $\sqrt{w/t}$ represents the volatility corresponding to a value of total variance. To help put the results of the WRMSE for both eSSVI and SVI in perspective, we also include a very naive linear model for the total implied variance. This model takes the form $w(k) = \max(ak + bt + c, 0)$, where the parameters a, b, and c are determined by minimizing an expression similar to (7). The results shown in Table 3 are typical for many underlying assets/indices and dates: for a reasonable loss in fitting error for the eSSVI compared to SVI, we gain the advantage of a surface free of static arbitrage.

	eSSVI	SVI	Linear
WRMSE	0.00958	0.00315	.07305

Table 1: WRMSE for different implied volatility surface models for SPX on March 8, 2018.

We can interpolate and extrapolate any of our models beyond the available market data on a given date [1]. Therefore, we can compute a time series for the implied volatility surface value over a range of dates for a fixed value of maturity and moneyness. This gives us some sense of how the model performs during stressed conditions. Figure 3 shows a time series for the calendar year 2008 for SPX that is ATM with 45 days to maturity.

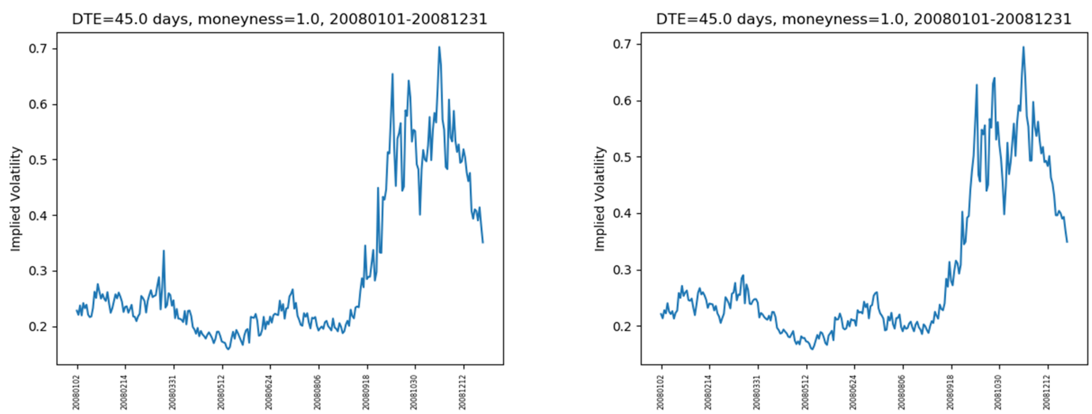

Figure 2: ATM, 45 days to maturity time series for calendar year 2008 for SPX. eSSVI is on the left, SVI is on the right.

Perhaps not surprisingly, both models yield very similar results at this moneyness value. However, we see a drastic difference when the moneyness is increased to 1.2, see Figure 3. The eSSVI model yields reasonable arbitrage-free results for the entire surface, whereas the feasibility of the SVI model quickly erodes from ATM.

[1] See Section 4 for how this is accomplished for the eSSVI model. For the SVI, we use a monotonically increasing interpolation with respect to maturity.

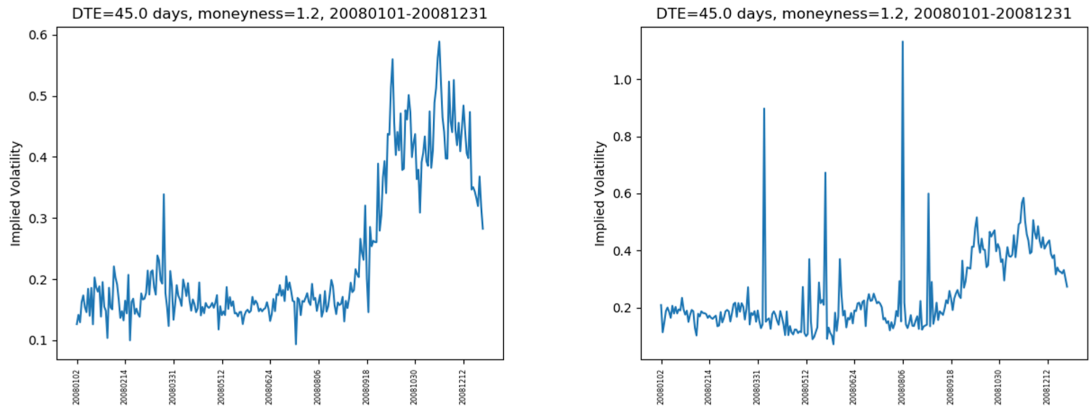

Figure 3: Moneyness = 1.2, 45 days to maturity time series for calendar year 2008 for SPX. eSSVI is on the left, SVI is on the right.

4 Interpolation and extrapolation

Suppose that we have calibrated (1) to market data at each available maturity such that inequalities (2)-(6) are satisfied. In other words, for each maturity $t_1 < t_2 < \cdots < t_n$ where market data exists, we have a set of parameters $(\theta_i, \rho_i, \psi_i)$ that allow us to compute a total variance for any strike using the model in (1). To compute total variance at other maturities, we need to be able to interpolate and extrapolate.

For interpolation, let $t_\lambda = (1-\lambda)t_i + \lambda t_{i+1}$ for some number λ such that $0 \leq \lambda \leq 1$. To this maturity t_λ we assign parameters as follows

$$\begin{aligned} \theta_\lambda &= (1-\lambda)\theta_i + \lambda\theta_{i+1} \\ \psi_\lambda &= (1-\lambda)\psi_i + \lambda\psi_{i+1} \\ \rho_\lambda &= \frac{(1-\lambda)\rho_i\psi_i + \lambda\rho_{i+1}\psi_{i+1}}{\psi_\lambda}. \end{aligned} \tag{8}$$

Total variance for any strike at maturity t_λ can then be computed using these parameters in (1). Note that the parameters θ_λ and ψ_λ are linearly interpolated, but ρ_λ is not.

For extrapolation in the maturity interval $[0, t_1]$, let $t_\lambda = \lambda t_1$ with $0 \leq \lambda \leq 1$. Here we set

$$\begin{aligned} \theta_\lambda &= \lambda\theta_1 \\ \psi_\lambda &= \lambda\psi_1 \\ \rho_\lambda &= \rho_1. \end{aligned} \tag{9}$$

So in this case θ_λ and ψ_λ are again linearly interpolated, and ρ_λ is held constant.

For extrapolation beyond t_n, let t_λ be any maturity such that $t_\lambda > t_n$. Set

$$\begin{aligned} \theta_\lambda &= M(t_\lambda - t_n) + \hat{\theta}_n \\ \psi_\lambda &= \psi_n \\ \rho_\lambda &= \rho_n. \end{aligned} \tag{10}$$

Here $\hat{\theta}_i$ is the ATM total variance predicted by (1) using the parameters $(\theta_i, \rho_i, \psi_i)$ and $k = 0$, and M is the slope obtained from linear regression on the set of points $\{(t_i, \hat{\theta}_i)\}$ for $i = 1, \ldots, n$.

Finally, we note that this interpolation/extrapolation scheme is not chosen arbitrarily. See [1] for proof that the resulting surface is free of butterfly- and calendar-spread arbitrage. Figure 4 shows the implied volatility surface for SPX on March 7, 2018. The smallest and largest maturities with available market data are 7 and 1017 days respectively, the second and third panels zoom in on these extrapolation transition points.

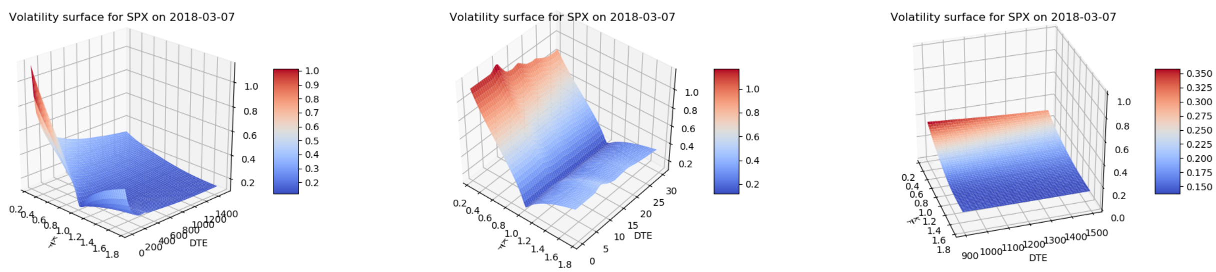

Figure 4: eSSVI implied volatility surface for SPX on March 7, 2018.

A Data Filtering

The following filtering rules are applied to the following exchanges (both equity and index securities):

Filters	ASX	BOM	DMI	EUR	KRX	LIF	MOD	MRV	OMX
No in-the-money	✓	✓	✓	✓	✓	✓	✓	✓	✓
No Adjusted Options	✓	✓	✓	✓	✓	✓	✓	✓	✓
No IV = 0 or NA	✓	✓	✓	✓	✓	✓	✓	✓	✓
No Secondary Exchange	✓	✓	✓	✓	✓	✓	✓	✓	✓
No European Style Equity Options	✓		✓	✓	✓	✓	✓	✓	✓
No Options with Bid/Ask = 0									✓
No options where last settlement date $\neq$ requested date						✓			
No options with Settlement price = 0 or NA	✓	✓	✓	✓	✓	✓	✓	✓	
No options with contract size $\neq$ 100									
No options where bid/ask % spread > 5%									
No options where contract size $\neq$ mode	✓	✓	✓	✓	✓	✓	✓	✓	✓

Filters	OSE	OSL	TFEX	USA	WAR	HKO	TAE	EMD	BUD
No in-the-money	✓	✓	✓	✓	✓	✓	✓	✓	✓
No Adjusted Options	✓	✓	✓	✓	✓	✓	✓	✓	✓
No IV = 0 or NA	✓	✓	✓	✓	✓	✓	✓	✓	✓
No Secondary Exchange	✓	✓	✓	✓	✓	✓	✓	✓	✓
No European Style Equity Options	✓	✓	✓	✓	✓	✓	✓	✓	✓
No Options with Bid/Ask = 0		✓		✓					
No options where last settlement date $\neq$ requested date							✓		
No options with Settlement price = 0 or NA	✓		✓		✓	✓	✓	✓	✓
No options with contract size $\neq$ 100				✓					
No options where bid/ask % spread > 5%				✓					
No options where contract size $\neq$ mode	✓	✓	✓		✓	✓	✓	✓	✓

B Dividend Methodologies

The dividend methodology used when calculating the forward price is determined by both the option type and the data available on the underlying ID. For Equity Options, discrete dividends will be used. For Index Options, a continuous dividend yield will be used.

B.1 Discrete dividends

When calculating dividend projections, we look at dividends paid over the past 12 months and any announced dividends available up to the option's expiration date. Any non-regular dividends are excluded.

When calculating on 10/9/2019 for an option that expired 10/23/2020, the following dividends are fetched:

Dividend amount	**Ex-date**	**Type**	**Category**
\$0.54	11/13/2018	Final	Historical
\$0.25	3/10/2019	Interim	Historical
\$0.26	7/10/2019	Interim	Historical
\$0.55	10/20/2019	Final	Announced

With the array of dividends, we project that each dividend ex-date over that period will have a corresponding dividend of the same type the following year. We check to see if a projected dividend is within ± 28 days of an actual or announced dividend (or ± 10 days for monthly dividends). If there is, we throw out the projection. We continue this up to the option's expiration. The amount of each dividend projection will come from the latest dividend of the same type.

When looking at final dividends, there was a dividend ex-date of 11/13/2018. We project a dividend will have an ex-date on 11/13/2019, with an amount of $0.55 (the most recent dividend payment of the final type). However, since the 11/13/2019 projection is within 28 days of the 10/20/2019 announced dividend, the projection is thrown out. The 3/10/2019 and 7/10/2019 dividends are projected forward annually.

Dividend amount	**Ex-date**	**Type**	**Category**
$0.54	11/13/2018	Final	Historical
$0.25	3/10/2019	Interim	Historical
$0.26	7/10/2019	Interim	Historical
$0.55	10/20/2019	Final	Announced
$0.55	11/13/2019	Final	Thrown out projection
$0.26	3/10/2020	Interim	Projected
$0.26	7/10/2020	Interim	Projected
$0.55	10/20/2020	Final	Projected

The final set of dividends included in the calculation on 10/9/2019 for 10/23/2020 expiration is:

Dividend amount	**Ex-date**	**Type**	**Category**
$0.55	10/20/2019	Final	Announced
$0.26	3/10/2020	Interim	Projected
$0.26	7/10/2020	Interim	Projected
$0.55	10/20/2020	Final	Projected

For an option expiring on 8/21/2021, the dividends included would be:

Dividend amount	**Ex-date**	**Type**	**Category**
$0.55	10/20/2019	Final	Announced
$0.26	3/10/2020	Interim	Projected
$0.26	7/10/2020	Interim	Projected
$0.55	10/20/2020	Final	Projected
$0.26	3/10/2021	Interim	Projected
$0.26	7/10/2021	Interim	Projected

In the case of Franked/Unfranked dividends, these are summed together and treated as a single dividend.

B.2 Continuous dividends

For indices, a continuous dividend yield is applied. When settlement prices are available, an implied dividend yield will be calculated for each maturity. The implied dividend yield is calculated for each put-call option pair in the filtered universe. The implied dividend yield is calculated as:

$$q_{\text{imp}} = \frac{\ln\left(\frac{(\text{Call Settle}) - (\text{Put Settle}) + Ke^{-rt}}{s}\right)}{t}.$$

Once the implied dividend yield is calculated for each put-call pair, the average value is calculated and used when calculating the forward price for that maturity. In the case where the average implied dividend yield is less than zero, no dividend yield is used.

References

[1] P. Cohort, J. Corbetta, C. Martini, and I. Laachir, *Robust calibration and arbitrage-free interpolation of SSVI slices*, arXiv, https://arxiv.org/abs/1804.04924, (2018).

[2] J. Gatheral, *The volatility surface: a practitioner's guide*, Wiley, New Jersey, (2006).

[3] J. Gatheral and A. Jacquier, *Convergence of Heston to SVI*, Quantitative Finance, vol 11, pp. 1129–1132, (2011).

[4] J. Gatheral and A. Jacquier, *Arbitrage-free SVI volatility surfaces*, Quantitative Finance, vol 14, pp. 59–71, (2014).

[5] S. Hendriks and C. Martini, *The extended SSVI volatility surface*, SSRN, https://ssrn.com/abstract=2971502, (2017).

[6] R. Lee, *The moment formula for implied volatility at extreme strikes*, Mathematical Finance, vol 14, pp. 469–480, (2004).

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