The Voting Algorithm Implements Consensus

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To say what it means for the $Voting$ algorithm to implement the $Consensus$ spec, we have to say what it means for the algorithm to choose a value.

More precisely, we have to say how the algorithm implements the variable $chosen$ of the spec.

This is stated in the definition of the expression $chosen$ in module $Voting$.

$$VotedFor(a, b, v) \triangleq \langle b, v \rangle \in votes[a]$$

$$VotedFor(a,\ b,\ v)\ \triangleq\ \langle b,\ v\rangle \in votes[a]$$

True iff acceptor a has voted for value v in ballot b.

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Consider any behavior of the $Voting$ algorithm.

Consider the value of $chosen$ in each state.

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Those values of $chosen$ should produce a behavior allowed by the $Consensus$ specification.

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But that's absurd.

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A behavior of the $Voting$ algorithm has lots of steps.
The $Consensus$ spec only allows a single step.

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The display then changes to 1, then to 2, and so on.

We go into the store and a salesman shows us a clock
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We could specify such a clock like this:

When the clock is plugged in, the display shows 12.

The display then changes to 1, then to 2, and so on.

We go into the store and a salesman shows us a clock that displays the hour and also the minute.

Since we didn't specify that the clock doesn't display the minute, most people would say the hour-minute clock satisfies our spec.

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A spec of an hour-minute clock talks about a different universe that contains an hour display and a minute display.

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Comparing the two specs isn't so simple.

If Math was Invented by Computer Scientists

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Theorem 1. If x is an integer, then $x + 1 > x$.

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You'd like to use Theorem 1 to prove Theorem 2.

But you couldn't because Theorem 1 is about a universe containing only one integer x, while Theorem 2 is about a different universe that contains the two integers x and y.

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That universe contains the variables x , y

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Theorem 1. If x is an integer, then $x + 1 > x$.

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are about the same universe.

That universe contains the variables x, y, z, q, α, β, $maxBal$, $chosen$, $\ldots$

Those theorems only say something about two of those variables.

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A spec of an hour clock doesn't describe a universe containing only a single variable $hour$.

It describes a universe containing infinitely many variables.

A state is an assignment of values to all those variables.

The hour clock's spec says nothing about the values of any variable other than $hour$.

When we say that the hour clock just has this single behavior:

$$\left[hour = 12\right] \longrightarrow \left[hour = 1\right] \longrightarrow \left[hour = 2\right] \longrightarrow \cdots$$

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we mean that it allows infinitely many behaviors, such as:

$$\begin{bmatrix} hour & = 12 \\ chosen & = \{\,\} \\ maxBal & = -72 \\ min & = 32 \\ x & = \sqrt{7} \\ & \vdots \end{bmatrix} \longrightarrow \begin{bmatrix} hour & = 1 \\ chosen & = \{0\} \\ maxBal & = -7 \\ min & = 16 \\ x & = \sqrt{-1} \\ & \vdots \end{bmatrix} \longrightarrow \begin{bmatrix} hour & = 2 \\ chosen & = \{\,\} \\ maxBal & = 1/2 \\ min & = \langle 1, 2 \rangle \\ x & = \sqrt{-1} \\ & \vdots \end{bmatrix} \longrightarrow \cdots$$

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All behaviors in which $hour$ has these values

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All behaviors in which $hour$ has these values
and the other variables can have any values.

When we say that the hour clock just has this single behavior:

$$\left[hour = 12\right] \quad \longrightarrow \quad \left[hour = 1\right] \quad \longrightarrow \quad \left[hour = 2\right] \quad \longrightarrow \ldots$$

When we say that the hour clock just has this single behavior:

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we also mean that it allows **only** behaviors such as:

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in which the $Voting$ algorithm can change $maxBal$ or $votes$ only once an hour.

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in which the $Voting$ algorithm can change $maxBal$ or $votes$ only once an hour.

This is silly.

The spec of an hour clock should not allow just this behavior:

$$\big[hour = 12\big] \quad \longrightarrow \quad \big[hour = 1\big] \quad \longrightarrow \quad \big[hour = 2\big] \quad \longrightarrow \; \ldots$$

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$$\left[hour = 12 \right] \longrightarrow \left[hour = 1 \right] \longrightarrow \left[hour = 2 \right] \longrightarrow \cdots$$

It should also allow behaviors such as:

$$\left[hour = 12 \right] \longrightarrow \left[hour = 12 \right] \longrightarrow \left[hour = 1 \right] \longrightarrow \left[hour = 1 \right] \longrightarrow$$

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stuttering steps

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And it does.

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$$Spec \triangleq Init \land \Box[Next]_{chosen}$$

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Now we find out what this is all about.

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it allows stuttering steps that leave $chosen$ unchanged.

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So the $Voting$ spec allows stuttering steps.

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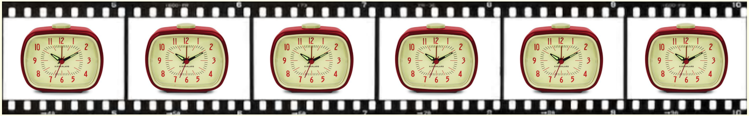
All TLA$^+$ specs allow stuttering steps.

Think of a state in a behavior as a frame in a movie of the system



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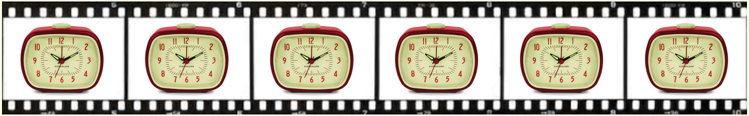

where the camera ran at an arbitrarily changing rate.

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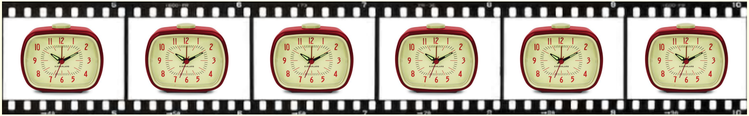

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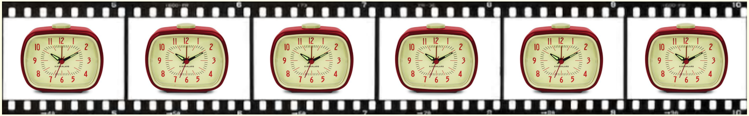

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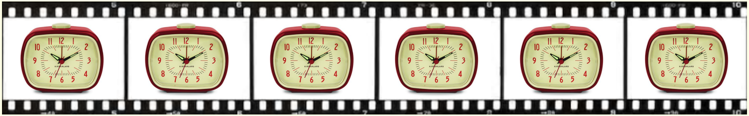

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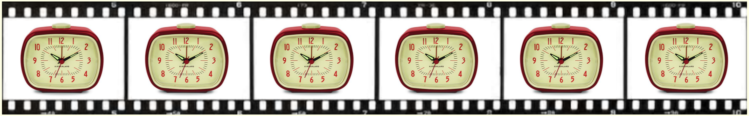

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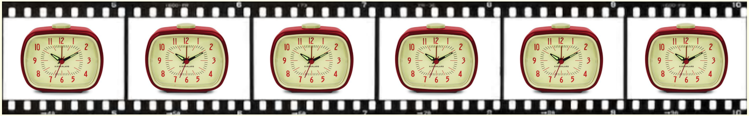

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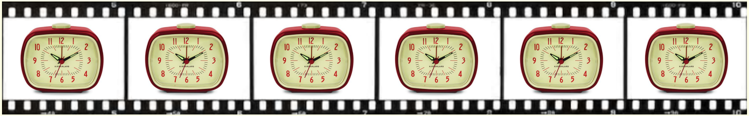

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We needn't do that because Paxos is an asynchronous algorithm.

We now return to our goal:

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$$\begin{aligned} & ChosenAt(b, v) \triangleq \\ & \quad \exists Q \in Quorum : \forall a \in Q : VotedFor(a, b, v) \end{aligned}$$

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$$chosen \triangleq \{v \in Value : \exists\, b \in Ballot : ChosenAt(b, v)\}$$

We have to show that for any behavior of the $Voting$ algorithm:

$$\begin{bmatrix} votes & = \cdots \\ maxBal & = \cdots \end{bmatrix} \longrightarrow \begin{bmatrix} votes & = \cdots \\ maxBal & = \cdots \end{bmatrix} \longrightarrow \begin{bmatrix} votes & = \cdots \\ maxBal & = \cdots \end{bmatrix} \longrightarrow \cdots$$

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If we take the values of $chosen$ in each state
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we get a behavior that satisfies the $Consensus$ spec.

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This condition is equivalent to the condition that the original behavior satisfies the formula obtained by substituting the definition of $chosen$ in $Voting$ for the variable $chosen$ in formula $Spec$ of $Consensus$.

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I don't expect you to see why these two conditions are equivalent.

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Let's call this formula $Spec_C^{sub}$.

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This condition is equivalent to the condition that the original behavior satisfies $Spec_C^{sub}$.

We have to show that for any behavior of the $Voting$ algorithm:

$$\begin{bmatrix} votes & = \cdots \\ maxBal & = \cdots \end{bmatrix} \longrightarrow \begin{bmatrix} votes & = \cdots \\ maxBal & = \cdots \end{bmatrix} \longrightarrow \begin{bmatrix} votes & = \cdots \\ maxBal & = \cdots \end{bmatrix} \longrightarrow \cdots$$

$$\begin{bmatrix} chosen = \cdots \end{bmatrix} \longrightarrow \begin{bmatrix} chosen = \cdots \end{bmatrix} \longrightarrow \begin{bmatrix} chosen = \cdots \end{bmatrix} \longrightarrow \cdots$$

If we take the values of $chosen$ in each state
and assign them to the variable $chosen$
we get a behavior that satisfies the $Consensus$ spec.

This condition is equivalent to the condition that the original behavior satisfies $Spec_C^{sub}$.

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We have to show:

Any behavior of the $Voting$ algorithm satisfies $Spec_C^{sub}$.

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Let $Spec_V$ be formula $Spec$ of module $Voting$.

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THEOREM $Spec_V \;\Rightarrow\; Spec_C^{sub}$

We have to show:

~~Any behavior satisfying $Spec_V$ satisfies $Spec_C^{sub}$~~.

THEOREM $Spec_V \Rightarrow Spec_C^{sub}$

Let's now see how that theorem is written in TLA$^+$.

THEOREM $Spec_V \Rightarrow Spec_C^{sub}$

THEOREM $Spec_V \Rightarrow Spec_C^{sub}$

We will write the theorem in module $Voting$

THEOREM $Spec_V \Rightarrow Spec_C^{sub}$

We will write the theorem in module $Voting$, so this is just $Spec$.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

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THEOREM $Spec \Rightarrow Spec_C^{sub}$

We will write the theorem in module $Voting$, so this is just $Spec$.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

To write $Spec_C^{sub}$, module $Voting$ must import the definition of $Spec$ from module $Consensus$.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

INSTANCE $Consensus$

This imports all the definitions from $Consensus$.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

INSTANCE $Consensus$

This imports all the definitions from $Consensus$.

But to prevent name clashes, the names of the imported definitions must be changed.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

This imports all the definitions from $Consensus$ with their names prefixed by $C!$.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

This imports all the definitions from $Consensus$ with their names prefixed by $C!$.

For example $Next$ is imported as $C!Next$.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

The following kinds of symbols appear in module $Consensus$:

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

The following kinds of symbols appear in module $Consensus$:

– TLA^+ primitive operators

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

The following kinds of symbols appear in module $Consensus$:

– TLA$^+$ primitive operators such as $\land \quad \subseteq \quad \Box$

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

The following kinds of symbols appear in module $Consensus$:

– TLA$^+$ primitive operators such as $\land \quad \subseteq \quad \Box$

They are meaningful in module $Voting$.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

The following kinds of symbols appear in module $Consensus$:

– TLA$^+$ primitive operators

– Defined operators

THEOREM $Spec \Rightarrow Spec^{sub}_{C}$

$C \triangleq$ INSTANCE $Consensus$

The following kinds of symbols appear in module $Consensus$:

- TLA$^+$ primitive operators
- Defined operators such as $Next$ Inv

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

The following kinds of symbols appear in module $Consensus$:

– TLA$^+$ primitive operators

– Defined operators such as $Next$ Inv $IsFiniteSet$ $\leq$

THEOREM $Spec \Rightarrow Spec_{C}^{sub}$

$C \triangleq$ INSTANCE $Consensus$

The following kinds of symbols appear in module $Consensus$:

- TLA^{+} primitive operators
- Defined operators such as $Next$ Inv $IsFiniteSet$ $\leq$

imported with EXTENDS

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

The following kinds of symbols appear in module $Consensus$:

- TLA$^+$ primitive operators
- Defined operators such as $Next$ Inv $IsFiniteSet$ $\leq$
  They are imported (renamed) into $Voting$ with their definitions.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

The following kinds of symbols appear in module $Consensus$:

- TLA$^+$ primitive operators
- Defined operators
- The declared symbols $Value$ and $chosen$.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

The following kinds of symbols appear in module $Consensus$:

– TLA$^+$ primitive operators

– Defined operators

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  Those symbols have meanings in module $Voting$.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

The following kinds of symbols appear in module $Consensus$:

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Those symbols have meanings in module $Voting$.

But how do we know those meanings are related to their meanings in module $Consensus$?

THEOREM $Spec \Rightarrow Spec_{C}^{sub}$

$C \triangleq$ INSTANCE $Consensus$

The following kinds of symbols appear in module $Consensus$:

– TLA^{+} primitive operators

– Defined operators

– The declared symbols $Value$ and $chosen$.

Those symbols have meanings in module $Voting$.

But how do we know those meanings are related to their meanings in module $Consensus$?

We have to say what expressions must be substituted for them.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

WITH $Value \leftarrow \quad , \; chosen \leftarrow$

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$

WITH $Value \leftarrow Value,\ chosen \leftarrow$

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$
WITH $Value \leftarrow Value, \; chosen \leftarrow chosen$

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$
WITH $\boxed{Value} \leftarrow Value$, $\boxed{chosen} \leftarrow chosen$

These are the declared symbols of module $Consensus$.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$
WITH $Value \leftarrow \boxed{Value}$, $chosen \leftarrow \boxed{chosen}$

These are the declared symbols of module $Consensus$.

The are expressions of module $Voting$.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$
WITH $Value \leftarrow Value,\ chosen \leftarrow chosen$

This symbol is defined in $Voting$

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$
WITH $Value \leftarrow Value,\ chosen \leftarrow chosen$

This symbol is defined in $Voting$ by

$$chosen \triangleq \{v \in Value : \exists\, b \in Ballot : ChosenAt(b, v)\}$$

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$
WITH $Value \leftarrow Value,\ chosen \leftarrow \boxed{chosen}$

This symbol is defined in $Voting$ by

$$chosen \triangleq \{v \in Value : \exists\, b \in Ballot : ChosenAt(b, v)\}$$

We can replace it

THEOREM $Spec \Rightarrow Spec_{C}^{sub}$

$C \triangleq$ INSTANCE $Consensus$
WITH $Value \leftarrow Value,\ chosen \leftarrow \boxed{chosen}$

This symbol is defined in *Voting* by

$$chosen \triangleq \boxed{\{v \in Value : \exists\, b \in Ballot : ChosenAt(b,\ v)\}}$$

We can replace it by this

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$C \triangleq$ INSTANCE $Consensus$
WITH $Value \leftarrow Value,\ chosen \leftarrow \boxed{chosen}$

This symbol is defined in $Voting$ by

$$chosen \triangleq \boxed{\{v \in Value : \exists\, b \in Ballot : ChosenAt(b, v)\}}$$

We can replace it by this, since the two are equivalent in module $Voting$.

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$$C \triangleq \text{INSTANCE } Consensus$$
$$\qquad \text{WITH } Value \leftarrow Value,\ chosen \leftarrow chosen$$

THEOREM $Spec \Rightarrow Spec_C^{sub}$

$$C \triangleq \text{INSTANCE } Consensus \\ \quad \text{WITH } Value \leftarrow Value,\ chosen \leftarrow chosen$$

So we can now write $Spec_C^{sub}$

THEOREM $Spec \Rightarrow C!Spec$

$$C \triangleq \text{INSTANCE } Consensus$$
$$\quad \text{WITH } Value \leftarrow Value,\ chosen \leftarrow chosen$$

So we can now write $Spec_C^{sub}$ as $C!Spec$.

THEOREM $Spec \Rightarrow C!Spec$

$$C \triangleq \text{INSTANCE } Consensus \\ \quad \text{WITH } Value \leftarrow Value,\ chosen \leftarrow chosen$$

$$C \triangleq \text{INSTANCE } Consensus$$
$$\text{WITH } Value \leftarrow Value,\ chosen \leftarrow chosen$$

$$\text{THEOREM } Spec \Rightarrow C!Spec$$

$$C \triangleq \text{INSTANCE } Consensus$$
$$\qquad\qquad \text{WITH } Value \leftarrow Value, \; chosen \leftarrow chosen$$

$$\text{THEOREM} \quad Spec \;\Rightarrow\; C!Spec$$

Of course, it has to go after the INSTANCE statement.

$$
\begin{array}{l}
C \;\triangleq\; \textsf{INSTANCE}\ \mathit{Consensus} \\
\qquad\qquad\quad \textsf{WITH}\ \ \mathit{Value} \leftarrow \mathit{Value},\ \mathit{chosen} \leftarrow \mathit{chosen}
\end{array}
$$

We can make one final simplification.

$$C \triangleq \text{INSTANCE } Consensus$$
$$\text{WITH } Value \leftarrow Value,\ \boxed{chosen} \leftarrow \boxed{chosen}$$

We can make one final simplification.

When we substitute the same symbol for a symbol

$$
\begin{aligned}
C \;\triangleq\; & \textsf{INSTANCE}\ \mathit{Consensus} \\
& \qquad \textsf{WITH}\ \ \mathit{Value} \leftarrow \mathit{Value}
\end{aligned}
$$

We can make one final simplification.

When we substitute the same symbol for a symbol we can omit that WITH clause.

$C \triangleq$ INSTANCE $Consensus$
WITH

We can make one final simplification.

When we substitute the same symbol for a symbol we can omit that WITH clause.

$C \triangleq$ INSTANCE $Consensus$

We can make one final simplification.

When we substitute the same symbol for a symbol we can omit that WITH clause.

$$C \triangleq \text{INSTANCE } Consensus$$

$$\text{THEOREM } \; Spec \Rightarrow C!Spec$$

$C \triangleq$ INSTANCE $Consensus$

THEOREM $Spec \Rightarrow C!Spec$

This theorem asserts that algorithm $Voting$ implements the $Consensus$ spec under its definition of $consensus$.

THEOREM $Spec \Rightarrow C!Spec$

THEOREM $Spec \Rightarrow C!Spec$

The model checker can check this theorem.

THEOREM $Spec \;\Rightarrow\; C!Spec$

The model checker can check this theorem.

We can also prove it

THEOREM $Spec \Rightarrow C!Spec$

The model checker can check this theorem.

We can also prove it, using a few simple proof rules.

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The proof uses an invariant maintained by the algorithm that explains why it is correct.

THEOREM $Spec \Rightarrow C!Spec$

The model checker can check this theorem.

We can also prove it, using a few simple proof rules.

The proof uses an invariant maintained by the algorithm that explains why it is correct.

The invariant is defined in the $Voting$ module.

Safety and Liveness

Safety and Liveness

I said this was a behavior satisfying the $Consensus$ spec

$$\left[chosen = \{\,\}\right] \longrightarrow \left[chosen = \{42\}\right]$$

Safety and Liveness

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because the spec allowed no step from this state.

Safety and Liveness

I said this was a behavior satisfying the $Consensus$ spec

$$\left[chosen = \{\,\}\right] \longrightarrow \left[chosen = \{42\}\right] \longrightarrow \left[chosen = \{42\}\right] \longrightarrow \left[chosen = \{42\}\right] \longrightarrow \ldots$$

because the spec allowed no step from this state.

That was wrong because it allows these steps.

Safety and Liveness

$$\Big[chosen = \{\,\}\Big] \longrightarrow \Big[chosen = \{42\}\Big] \longrightarrow \Big[chosen = \{42\}\Big] \longrightarrow \Big[chosen = \{42\}\Big] \longrightarrow \ldots$$

The hour clock and everything else doesn't stop
just because the $Consensus$ "algorithm" stops.

Safety and Liveness

$$\Big[chosen = \{\,\}\Big] \longrightarrow \Big[chosen = \{42\}\Big] \longrightarrow \Big[chosen = \{42\}\Big] \longrightarrow \Big[chosen = \{42\}\Big] \longrightarrow \ldots$$

The hour clock and everything else doesn't stop just because the $Consensus$ "algorithm" stops.

Termination of a system execution is represented by a behavior ending in all stuttering steps.

Safety and Liveness

$$\left[chosen = \{\,\}\right] \longrightarrow \left[chosen = \{42\}\right] \longrightarrow \left[chosen = \{42\}\right] \longrightarrow \left[chosen = \{42\}\right] \longrightarrow \ldots$$

The hour clock and everything else doesn't stop
just because the $Consensus$ "algorithm" stops.

Termination of a system execution is represented by
a behavior ending in all stuttering steps.

This makes the math simpler.

The spec $Init \land \Box[Next]_{chosen}$ of module $Consensus$ allows these behaviors:

The spec $Init \wedge \Box[Next]_{chosen}$ of module $Consensus$ allows these behaviors:

$$\big[chosen = \{\,\}\big] \rightarrow \big[chosen = \{42\}\big] \rightarrow \big[chosen = \{42\}\big] \rightarrow \big[chosen = \{42\}\big] \rightarrow \ldots$$

The spec $Init \land \Box[Next]_{chosen}$ of module $Consensus$ allows these behaviors:

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$$\left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{42\}\right] \rightarrow \ldots$$

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$$[chosen = \{\}] \rightarrow [chosen = \{\}] \rightarrow [chosen = \{\}] \rightarrow [chosen = \{42\}] \rightarrow \cdots$$

$$[chosen = \{\}] \rightarrow [chosen = \{\}] \rightarrow [chosen = \{\}] \rightarrow [chosen = \{\}] \rightarrow \cdots$$

$$\vdots$$

The spec $Init \land \Box[Next]_{chosen}$ of module $Consensus$ allows these behaviors:

$$\big[chosen = \{\,\}\big] \rightarrow \big[chosen = \{42\}\big] \rightarrow \big[chosen = \{42\}\big] \rightarrow \big[chosen = \{42\}\big] \rightarrow \cdots$$

$$\big[chosen = \{\,\}\big] \rightarrow \big[chosen = \{\,\}\big] \rightarrow \big[chosen = \{42\}\big] \rightarrow \big[chosen = \{42\}\big] \rightarrow \cdots$$

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$$\big[chosen = \{\,\}\big] \rightarrow \big[chosen = \{\,\}\big] \rightarrow \big[chosen = \{\,\}\big] \rightarrow \big[chosen = \{\,\}\big] \rightarrow \cdots$$

$$\vdots$$

Behaviors that take an arbitrary number of stuttering steps before a value is chosen.

The spec $Init \wedge \Box[Next]_{chosen}$ of module $Consensus$ allows these behaviors:

$$\left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{42\}\right] \rightarrow \left[chosen = \{42\}\right] \rightarrow \left[chosen = \{42\}\right] \rightarrow \cdots$$

$$\left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{42\}\right] \rightarrow \left[chosen = \{42\}\right] \rightarrow \cdots$$

$$\left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{42\}\right] \rightarrow \cdots$$

$$\left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \cdots$$

$$\vdots$$

It also allows a behavior containing only stuttering steps

The spec $Init \land \Box[Next]_{chosen}$ of module $Consensus$ allows these behaviors:

$$\left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{42\}\right] \rightarrow \left[chosen = \{42\}\right] \rightarrow \left[chosen = \{42\}\right] \rightarrow \cdots$$

$$\left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{42\}\right] \rightarrow \left[chosen = \{42\}\right] \rightarrow \cdots$$

$$\left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{42\}\right] \rightarrow \cdots$$

$$\left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \left[chosen = \{\,\}\right] \rightarrow \cdots$$

$$\vdots$$

It also allows a behavior containing only stuttering steps

describing an execution that terminates without choosing a value.

$Init \wedge \Box[Next]_{chosen}$

$Init \;\wedge\; \Box[Next]_{chosen}$

This formula says what steps are allowed to occur.

$Init \land \Box[Next]_{chosen}$

This formula says what steps are allowed to occur.

It doesn’t say what steps must occur.

$$Init \land \Box[Next]_{chosen}$$

An assertion of what is allowed to happen is called a *safety* property.

$Init \wedge \Box[Next]_{chosen}$

An assertion of what is allowed to happen is called a *safety* property.

An assertion of what must happen is called a *liveness* property.

$Init \wedge \Box[Next]_{chosen}$

An assertion of what is allowed to happen is called a *safety* property.

An assertion of what must happen is called a *liveness* property.

To specify that a value must be chosen,

$Init \land \Box[Next]_{chosen} \;\; \land \;\; \ldots$

An assertion of what is allowed to happen is called a *safety* property.

An assertion of what must happen is called a *liveness* property.

To specify that a value must be chosen, we'd have to conjoin a liveness property to the spec.

$Init \land \Box[Next]_{chosen} \;\; \land \;\; \ldots$

An assertion of what is allowed to happen is called a *safety* property.

An assertion of what must happen is called a *liveness* property.

To specify that a value must be chosen, we'd have to conjoin a liveness property to the spec.

That's easy to do

$Init \land \Box[Next]_{chosen} \;\; \land \;\; \ldots$

An assertion of what is allowed to happen is called a *safety* property.

An assertion of what must happen is called a *liveness* property.

To specify that a value must be chosen, we'd have to conjoin a liveness property to the spec.

That's easy to do (TLA$^+$ is great for specifying liveness)

$Init \land \Box[Next]_{chosen} \;\; \land \;\; \ldots$

An assertion of what is allowed to happen is called a *safety* property.

An assertion of what must happen is called a *liveness* property.

To specify that a value must be chosen, we'd have to conjoin a liveness property to the spec.

That's easy to do, but I won't do it.

$Init \wedge \Box[Next]_{chosen}$

An assertion of what is allowed to happen is called a *safety* property.

An assertion of what must happen is called a *liveness* property.

To specify that a value must be chosen, we'd have to conjoin a liveness property to the spec.

That's easy to do, but I won't do it.

Adding the requirement that a value must be chosen produces a spec that we can't implement.

The FLP Theorem

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Impossibility of Distributed Consensus with One Faulty Process

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Michael J. **F**ischer, Nancy A. **L**ynch, and Michael S. **P**aterson

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The Paxos consensus algorithm is useful because

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It's impossible to implement consensus with the requirement that a value is eventually chosen in a distributed system in which any single process may fail by stopping.

The Paxos consensus algorithm is useful because

- It never chooses more than one value no matter how many processes stop.

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It's impossible to implement consensus with the requirement that a value is eventually chosen in a distributed system in which any single process may fail by stopping.

The Paxos consensus algorithm is useful because

– It never chooses more than one value no matter how many processes stop.

– It has a very high probability of choosing a value if not too many processes stop.