exploding dots

GLOBALMATHPROJECT

buzzmath

EKS

CHAPTER 6

EXPLODING DOTS
CHAPTER 6

ALL BASES, ALL AT ONCE

The first chapters of this story have taken us through much of grade school mathematics. Let's now head on to advanced high school algebra. Whoa!

But here's the thing: there is nothing to it. We've already done all the work.

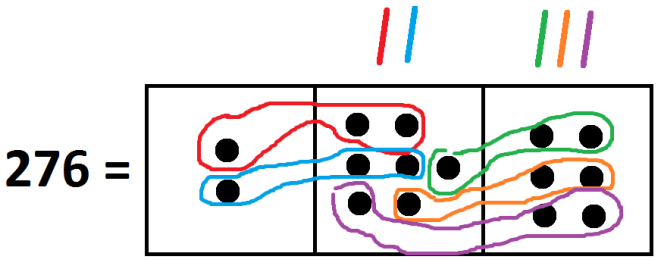
The only thing we have to realize is that there is nothing special about a $1 \leftarrow 10$ machine. We could do all of grade school arithmetic in a $1 \leftarrow 2$ machine if we wanted to, or a $1 \leftarrow 5$ machine, or even a $1 \leftarrow 37$ machine. The math doesn't care in which machine we do it. It is only us humans with a predilection for the number ten that draws us to the $1 \leftarrow 10$ machine.

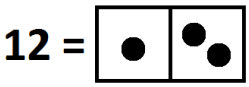
Let's now go through much of what we've done. But let's now do it in all possible machines, all at once!

Sounds crazy. But it is easy.

DIVISION IN ANY BASE

Here's the division problem $276 \div 12$ we did earlier in a $1 \leftarrow 10$ machine. We see the answer 23. Stare at this picture for a moment – it will soon sneak back up on us.





Let's now do the same division problem in another base. But the only tricky part is that I am not going to tell you which machine we are in! We could be in a $1 \leftarrow 10$ machine again, I am just not going to say.

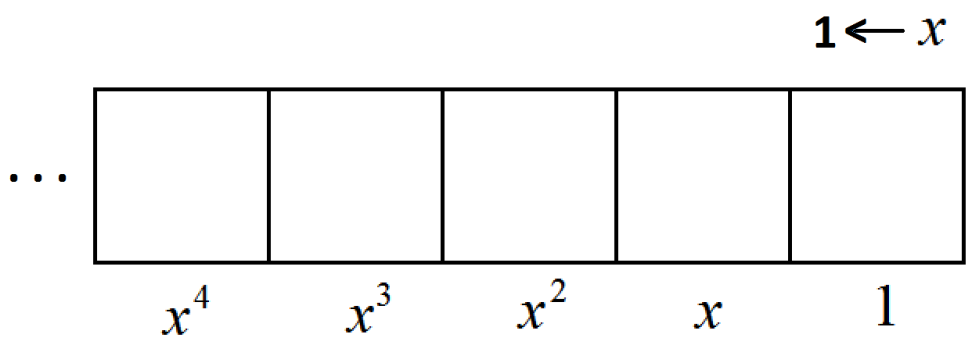
Maybe it will be in a $1 \leftarrow 2$, or a $1 \leftarrow 4$ machine or a $1 \leftarrow 13$ machine. You just won't know as I am not telling. It's the mood I am in!

Now, in high school algebra there seems to be a favorite letter of the alphabet to use for a quantity whose value you do not know. It's the letter x.

So let's work with an $1 \leftarrow x$ machine with the letter x representing some number whose actual value we do not know.

In a $1 \leftarrow 10$ machine the place values of the boxes are the powers of ten: , 10, 100, 1000, … .
In a $1 \leftarrow 2$ machine the place values of the boxes are the powers of two: 1, 2, 4, 8, 16, ….
And so on.

Thus, in a machine, the place values of the boxes will be the powers of x.



As a check, if I do tell you that x actually is 10 in my mind, then the powers 1, x, x^2, x^3, … match the numbers , 10, 100, 1000, … , which is correct for a $1 \leftarrow 10$ machine. If, instead, I tell you x is really 2 in my mind, then the powers 1, x, x^2, x^3, … match the numbers 1, 2, 4, 8, 16, …, which is correct for a $1 \leftarrow 2$ machine.

This $1 \leftarrow x$ machine really is representing all machines all at once!

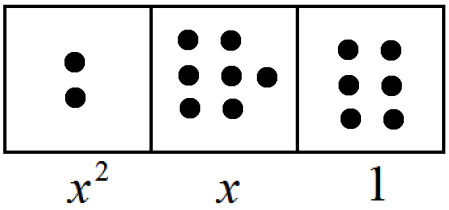
Okay. Out of the blue! Here's an advanced high school algebra problem.

Compute

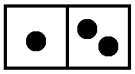
$$\boxed{\left(2x^2+7x+6\right)\div\left(x+2\right)\ .}$$

Can you figure out what this means on an $1 \leftarrow x$ machine? Try playing with this before reading on.

Here's what looks like in an $1 \leftarrow x$ machine. It's two x^2 s, seven x s, and six ones.

$2x^2 + 7x + 6 =$ 

And here's what $x+2$ looks like.

$x + 2 =$ 

The division problem $(2x^2 + 7x + 6) \div (x+2)$ is asking us to find copies of $x+2$ in the picture of $2x^2 + 7x + 6$.

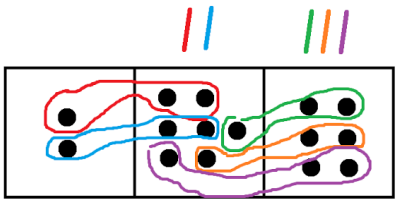
I see two copies of $x+2$ at the x level and three copies at the 1 level. The answer is $2x+3$.

Stare at the picture for . Does it look familiar?

We've just done a high school algebra problem as though it is a grade school arithmetic problem!

In a $1 \leftarrow 10$ machine.

$$276 \div 12$$
$$= 23$$



In a $1 \leftarrow x$ machine.

$$\left(2x^2 + 7x + 6\right) \div \left(x + 2\right)$$
$$= 2x + 3$$

SAME PICTURE!

What's going on?

Suppose I told you that really was 10 in my head all along. Then $2x^2 + 7x + 6$ is the number $2 \times 100 + 7 \times 10 + 6$, which is . And $x + 2$ is the number $10 + 2$, that is, . And so we computed $276 \div 12$. We got the answer , which is $2 \times 10 + 3 = 23$, if I am indeed now telling you that x is 10 .

So we did just repeat a grade-school arithmetic problem!

Aside: By the way, if I tell you that x was instead 2 , then

, which is ,

$x + 2 = 2 + 2$, which is 4 ,

and

, which is .

We just computed $28 \div 4 = 7$, which is correct!

Doing division in an machine is really doing an infinite number of division problems all in one hit. Whoa!

Try computing in an machine to get the answer $2x^2 + x + 3$.

(And if I tell you x is 10 in my mind, can you see that this matches $2556 \div 12 = 213$?)

In high school, numbers expressed in an $1 \leftarrow x$ machine are usually called *polynomials*. They are just

like numbers expressed in base 10, except now they are "numbers" expressed in base x. (And if someone tells you x is actually 10, then they really are base 10 numbers!)

Keeping this in mind makes so much of high school algebra so straightforward: it is a repeat of grade school base 10 arithmetic.

Here are some practice problems for you to try, it you like. My answers to them appear at the end of the chapter.

1. a) Compute $\left(2x^4+3x^3+5x^2+4x+1\right)\div\left(2x+1\right)$.

b) Compute $\left(x^4+3x^3+6x^2+5x+3\right)\div\left(x^2+x+1\right)$.

If I tell you x is actually 10 in both these problems what two division problems in ordinary arithmetic have you just computed?

2. Here's a polynomial division problem written in fraction notation. Can you do it? (Is there a slight difficulty to watch out for?)

$$\frac{x^4+2x^3+4x^2+6x+3}{x^2+3}$$

3. Show that $\left(x^4+4x^3+6x^2+4x+1\right)\div\left(x+1\right)$ equals x^3+3x^2+3x+1.

a) What is this saying for $x=10$?

b) What is this saying for $x=2$?

c) What is this saying for x equal to each of 3, 4, 5, 6, 7, 8, 9, and 11?

d) What is it saying for $x=0$?

e) What is it saying for $x=-1$?

A PROBLEM

Okay. Now that we are feeling really good about doing advanced algebra, I have a confession to make. I've been fooling you!

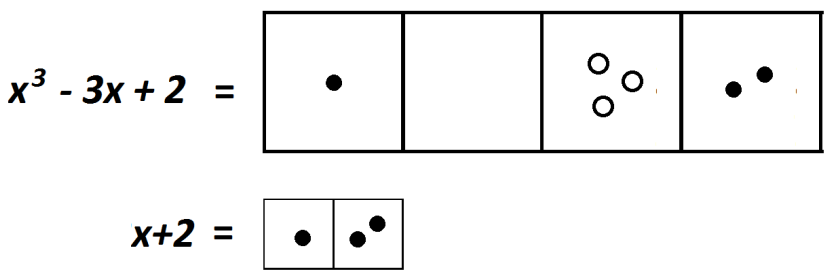
I've been choosing examples that are designed to be nice and to work out just beautifully. The truth is, this fabulous method of ours doesn't usually work so nicely.

Consider, for example,

$$\frac{x^3 - 3x + 2}{x + 2} \quad .$$

Do you see what I've been avoiding all this time? Yep. Negative numbers.

Here's what I see in an $1 \leftarrow x$ machine.



We are looking for one dot next to two dots in the picture of $x^3 - 3x + 2$. And I don't see any!

So what can we do, besides weep a little? Do you have any ideas?

It is tempting to say that we should just unexplode some dots. That's a brilliant idea! Except … we don't know a value for x and so don't know how many dots to draw when we unexplode. Bother!

We need some amazing flash of insight for something clever to do. Or maybe polynomial problems with negative numbers just can't be solved with this dots and boxes method.

What do you think? Any flashes of insight?

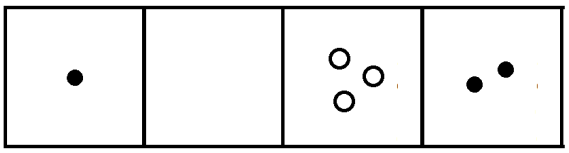
RESOLUTION

Here's the division problem we are stuck on.

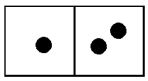
$$\frac{x^3 - 3x + 2}{x + 2}$$

And here is the picture for it again an $1 \leftarrow x$ machine.

x^3 - 3x + 2 =



x+2 =



We are looking for copies of $x + 2$, one dot next to two dots, anywhere in the picture of .
We don't see any.

And we can't unexplode dots to help us out as we don't know the value of . (We don't know how many dots to draw when we unexplode.)

The situation seems hopeless at present.

But I have a piece of advice for you, a general life lesson in fact. It's this.

> **IF THERE IS SOMETHING IN LIFE YOU WANT, MAKE IT HAPPEN!**
> **(And deal with the consequences.)**

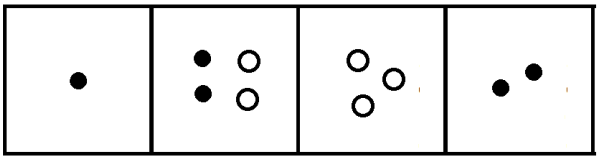
Right now, is there anything in life we want?

Look at that single dot way at the left. Wouldn't it be nice to have two dots in the box next to it to make a copy of $x + 2$?

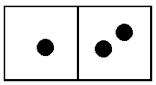
So let's just put two dots into that empty box! That's what I want, so let's make it happen!

But there are consequences: that box is meant to be empty. And in order to keep it empty, we can put in two antidots as well!

$x^3 - 3x + 2 =$



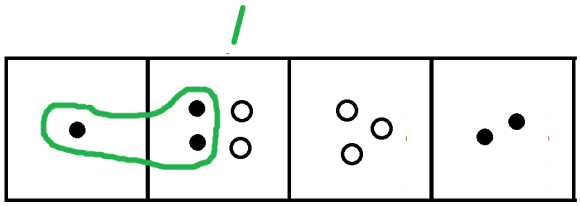
$x+2 =$



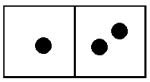
Brilliant!

We now have one copy of what we're looking for.

$x^3 - 3x + 2 =$



$x+2 =$



Is there anything else in life you want right now? Can you create another copy of $x+2$ anywhere?

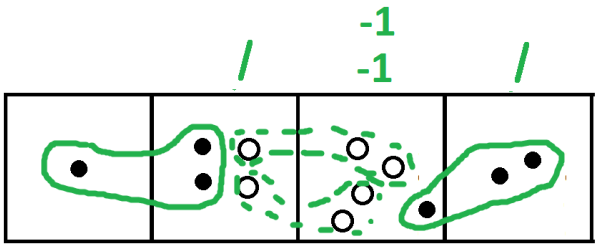
I'd personally like a dot to the left of the pair dots in the rightmost box. I am going to make it happen! I am going to insert a dot and antidot pair. Doing do finds me another copy of .

This is all well and good, but are we now stuck? Maybe this brilliant idea wasn't actually helpful in the end.

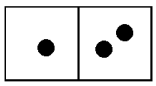
Stare at this picture for a while. Do you notice anything?

Look closely and we start to see copies of the exact opposite of what we're looking for! Instead of one dot next to two dots, there are copies of one antidot next to two antidots.

$x^3 - 3x + 2 =$



$x+2 =$



Whoa!

And how do we read the answer? We see that $\left(x^3 - 3x + 2\right) \div \left(x + 2\right)$ is $x^2 - 2x + 1$.

Fabulous!

So actually I was lying about fooling you. We can actually do all polynomial division problems with this dots and boxes method, even ones with negative numbers!

If you are looking for some practice problems, feel free to try these. Try them with pencil and paper, and then with the app perhaps. Answers, as usual, are at the end of this lesson.

4. Compute $\dfrac{x^3 - 3x^2 + 3x - 1}{x - 1}$.

5. Try .

6. If you can do this problem, , you can probably do any problem!

7. This one is crazy fun: $\dfrac{x^{10} - 1}{x^2 - 1}$.

Aside: Is there a way to conduct the dots and boxes approach with ease on paper? Rather than draw boxes and dots, can one work with tables of numbers that keep track of coefficients? (The word *synthetic*

is often used for algorithms one creates that are a step or two removed from that actual process at hand.)

REMAINDERS

It is just as easy to identify remainders in base x division problems as it is in base arithmetic.

Play with

in an machine. Can you see that it equals with a remainder of $2x-3$ yet to be divided by x^2-x+1 ?

People typically write this answer as follows:

$$\frac{4x^4-7x^3+9x^2-3x-1}{x^2-x+1}=4x^2-3x+2+\frac{2x-3}{x^2-x+1}.$$

Here are some practice problems if you would like to play some more with this idea.

8. Can you deduce what the answer to $\left(2x^2+7x+7\right)\div\left(x+2\right)$ is going to be before doing it?

9. Compute $\frac{x^4}{x^2-3}$.

10. Try this crazy one: .

If you do it with paper and pencil, you will find yourself trying to draw 84 dots at some point. Is it swift and easy just to write the number "84"? In fact, how about just writing numbers and not bother drawing any dots at all?

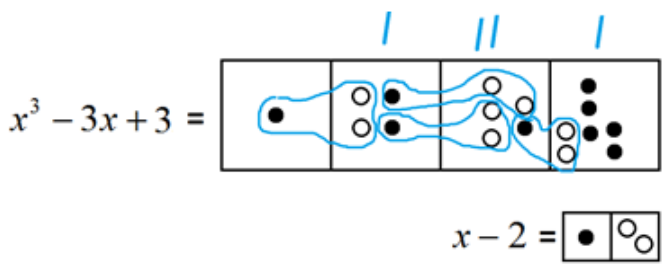
OPTIONAL: THE REMAINDER THEOREM

High school teachers have asked me if the dots and boxes approach can be used to explain the "Remainder Theorem." This optional section is for anyone interested in learning about the mathematics of this piece of extra-advanced polynomial algebra.

WARNING: This passage is not for the faint hearted!

Let's examine $\dfrac{x^3-3x+3}{x-2}$. This is the polynomial $p(x)=x^3-3x+3$ divided by the simple (linear) polynomial .

Here's what I get on the $1 \leftarrow x$ machine. (I had to add in some of dot/antidot pairs.) Check this!



We see that . There is a remainder of 5.

But let's look at the picture of carefully, taking note of the loops.

We see one loop at the level, two at the level, and one at the ones level. Plus we see a remainder of . As each loop represents the quantity , this means that

.

(This is one at the level, two at the x level, and one at the ones level, and .)

This shows that is a combination of s plus an extra .

.

That " " is standing out like a sore thumb. If you put in we get

.

In general, dividing a polynomial by a term of the form $x-h$ will give

where is a remainder. Putting $x=h$ shows that .

This is the Remainder Theorem for polynomials.

> *Dividing a polynomial $p(x)$ by a term $x-h$ gives a remainder that is a single number equal to , the value of the polynomial at $x=h$.*

People like this theorem because it shows that if $p(h)=0$ for some number , then $p(x)$ is an multiple of $x-h$. (The remainder is zero.) This gives the Factor Theorem for polynomials.

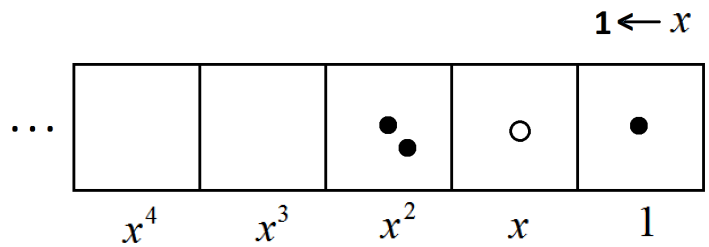
> *A polynomial p has a factor $x-h$ precisely when h is a zero of the polynomial, that is, precisely when .*

This is a big deal for people interested in factoring.

MULTIPLYING POLYNOMIALS

Can we multiply polynomials? You bet!

Here's the polynomial $2x^2 - x + 1$.



If we want to multiply this polynomial by 3 we just have to replace each dot and each antidot with three copies of it. (We want to triple all the quantities we see.)

We literally see that is .

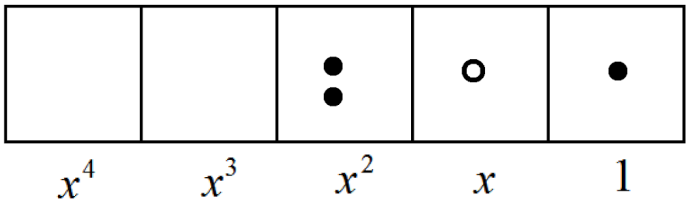
Suppose we wish to multiply by instead. This means we want the anti-version of tripling all the quantities we see. So each dot in the picture of $2x^2 - x + 1$ is to be replaced with three antidots and each antidot with three dots.

We have $-3\left(2x^2 - x + 1\right) = -6x^2 + 3x - 3$. We could also say that $-3\left(2x^2 - x + 1\right)$ is the anti-version of $3\left(2x^2 - x + 1\right)$.

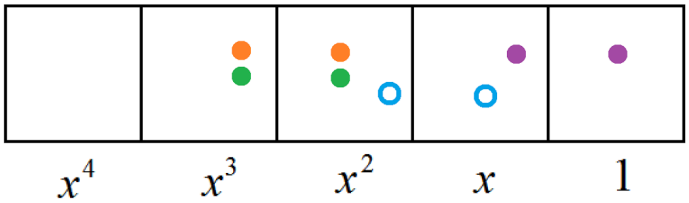
Now suppose we wish to multiply by $x+1$. Since $x+1$ looks like this

we need to replace each dot in the picture of with one-dot-and-one-dot, and each antidot with the anti-version of this, which is one-antidot-and-one-antidot. (This is now getting fun!)

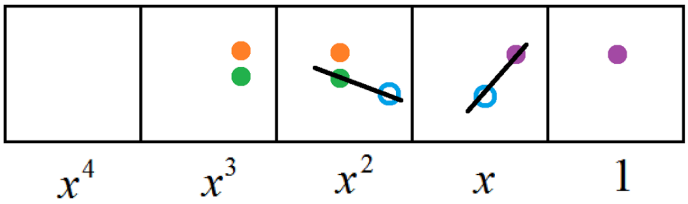
$2x^2 - x + 1 =$



$(x+1)\times(2x^2 - x + 1) =$

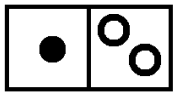


$=$



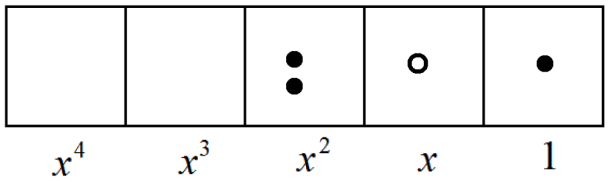
After some annihilations we see that $(x+1)\times(2x^2 - x + 1)$ equals $2x^3 + x^2 + 1$.

Now let's multiply $2x^2 - x + 1$ with $x - 2$, which looks like this.

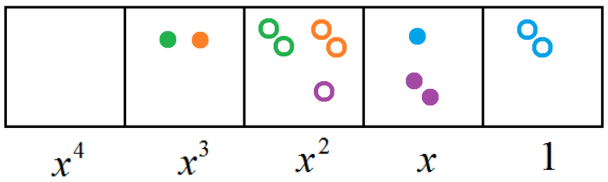


Each dot is to be replaced by one-dot-and-two-antidots, and each antidot with the opposite of this.

$2x^2 - x + 1 =$

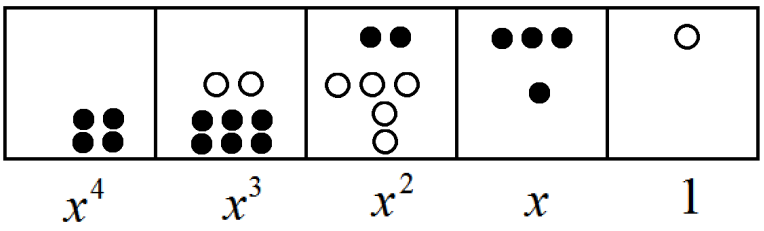


$(x-2)\times(2x^2 - x + 1) =$



We see $(x-2)(2x^2 - x + 1) = 2x^3 - 5x^2 + 3x - 2$.

Okay, you're turn. Try $2x^2 - x + 1$ times $2x^2 + 3x - 1$. Do you get this picture? (I've not colored it this time!) Do you see the answer ?



ADDING AND SUBTRACTING POLYNOMIALS

Adding and subtracting in base x is just like adding and subtracting in base 10. And it is easier in fact! Since we don't know the value of x we will never explode dots. That is, we never need to perform "carries" as one does in base arithmetic!

$$\begin{array}{rr} & 2x^2 + 8x - 5 \\ + & 9x^2 + 7x + 6 \\ \hline = & 11x^2 + 15x + 1 \end{array} \qquad \begin{array}{rr} & 10x^3 + 5x^2 - 7x + 3 \\ - & 3x^3 + 8x^2 + 5x - 2 \\ \hline = & 7x^3 - 3x^2 - 12x + 5 \end{array}$$

We can draw dots and boxes pictures of these in an $1 \leftarrow x$ machine if we like.

WILD EXPLORATIONS

Here are some "big question" investigations you might want to explore, or just think about. Have fun!

EXPLORATION 1: CAN WE EXPLAIN AN ARITHMETIC TRICK?

Here's an unusual way to divide by nine.

To compute $21203 \div 9$, say, read " 21203 " from left to right computing the partial sums of the digits along the way

2	**=2**
2+1	**=3**
2+1+2	**= 5**
2+1+2+0	**= 5**
2+1+2+0+3	**= 8**

and then read off the answer

$$21203 \div 9 = 2355 \ R \ 8 .$$

In the same way,

$$1033 \div 9 = 1 \mid 1+0 \mid 1+0+3 \mid \quad R \quad 1+0+3+3 \quad = 114 \ R \ 7$$

and

$$2222 \div 9 = 246 \ R \ 8 .$$

Can you explain why this trick works?

Here's the approach I might take: For the first example, draw a picture of in a $1 \leftarrow 10$, but think of nine as $10-1$. That is, look for copies of  in the picture.

EXPLORATION 2: CAN WE EXPLORE NUMBER THEORY?

Use an $1 \leftarrow x$ machine to compute each of the following

a) $\dfrac{x^2-1}{x-1}$ b) $\dfrac{x^3-1}{x-1}$ c) $\dfrac{x^6-1}{x-1}$ d) $\dfrac{x^{10}-1}{x-1}$

Can you now see that $\dfrac{x^{\text{number}}-1}{x-1}$ will always have a nice answer without a remainder?

Another way of saying this is that

$$x^{\text{number}}-1=(x-1)\times(\text{something}).$$

For example, you might have seen from part c) that $x^6-1=(x-1)(x^5+x^4+x^3+x^2+x+1)$. This means we can say, for example, that 17^6-1 is sure to be a multiple of ! How? Just choose $x=17$ in this formula to get

$$17^6-1=(17-1)\times(\text{something})=16\times(\text{something}).$$

e) Explain why $999^{100}-1$ must be a multiple of 998.

f) Can you explain why $2^{100}-1$ must be a multiple of 3, and a multiple of 15, and a multiple of 31 and a multiple of 1023? (Hint: $2^{100}=(2^2)^{50}=4^{50}$, and so on.)

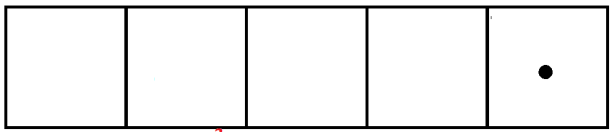
g) Is $x^{\text{number}}-1$ always a multiple of $x+1$? Sometimes, at least?

h) The number $2^{100}+1$ is not prime. It is a multiple of 17. Can you see how to prove this?

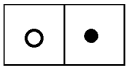
EXPLORATION 3: AN INFINITE ANSWER?

Here is a picture of the very simple polynomial 1 and the polynomial $1-x$.

1 = ···



1-x =



Can you compute $\frac{1}{1-x}$? Can you interpret the answer?

(We'll explore this example in the next chapter.)

SOLUTIONS

As promised, here are my solutions to the question posed.

Here are my answers.

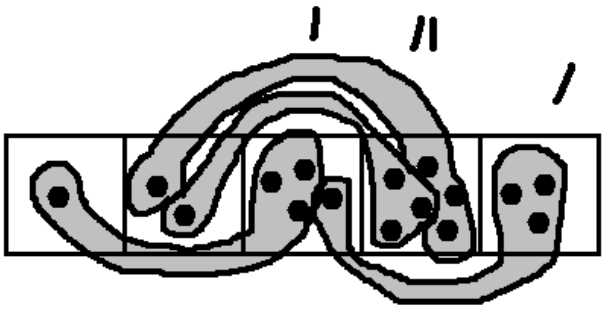
1.

a) $\left(2x^4+3x^3+5x^2+4x+1\right)\div\left(2x+1\right)=x^3+x^2+2x+1$

b) $\left(x^4+3x^3+6x^2+5x+3\right)\div\left(x^2+x+1\right)=x^2+2x+3$

And if x happens to be 10, we've just computed $23541\div 21=1121$ and $13653\div 111=123$.

2. We can do it. The answer is x^2+2x+1 .



3.

a) For $x=10$ it says $14641\div 11=1331$

b) For $x=2$ it says

c) For $x=3$ it says $256\div 4=64$

For $x=4$ it says

For $x=5$ it says $1296\div 6=216$

For it says

For $x=7$ it says

For $x=8$ it says $6561\div 9=729$

For $x = 9$ it says $10000 \div 10 = 1000$

For $x = 11$ it says $20736 \div 12 = 1728$

d) For $x = 0$ it says $1 \div 1 = 1$.

e) For it says $0 \div 0 = 0$. Hmm! That's fishy! (Can you have a 1←0 machine?)

4. .

5. $\dfrac{4x^3 - 14x^2 + 14x - 3}{2x - 3} = 2x^2 - 4x + 1$.

6. .

7. $\dfrac{x^{10} - 1}{x^2 - 1} = x^8 + x^6 + x^4 + x^2 + 1$.

8. We know that $(2x^2 + 7x + 6) \div (x + 2) = 2x + 3$ so I bet $(2x^2 + 7x + 7) \div (x + 2)$ turns out to be $2x + 3 + \dfrac{1}{x + 2}$. Does it?

9. $\dfrac{x^4}{x^2 - 3} = x^2 + 3 + \dfrac{9}{x^2 - 3}$.

10. $5x^2 - 2x + 21 + \dfrac{-14x^2 + 82x - 14}{x^3 - 4x + 1}$.