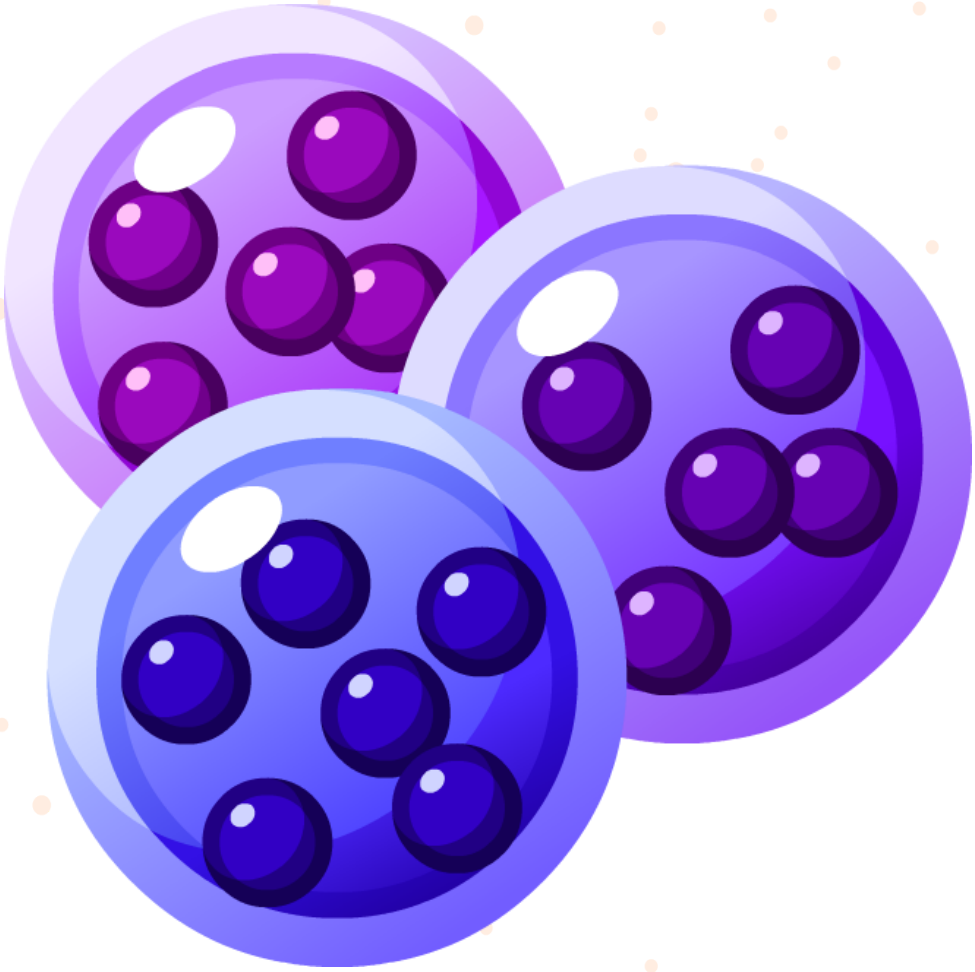
exploding dots

GLOBALMATHPROJECT

buzzmath



OBELUS

CHAPTER 5

EXPLODING DOTS
CHAPTER 5

DIVISION

Addition, subtraction, multiplication. Now it is time for division.

Here's an example of a division problem: *Compute* $276 \div 12$.

And here's a horrible way to solve it: *Draw a picture of* 276 *dots on a page and then circle groups of twelve dots. You will see, after about an hour, that there are* 23 *groups of twelve in a picture of* 276 .

Here's a great way to solve it: *Draw a picture of* 276 *dots in a* $1 \leftarrow 10$ *machine and just see right away that there are* 23 *groups of twelve in it!*

Read and play on to see how we can do this!

Cool Fact: Did you that the division symbol $\div$ is called an *obelus*?

Getting Started

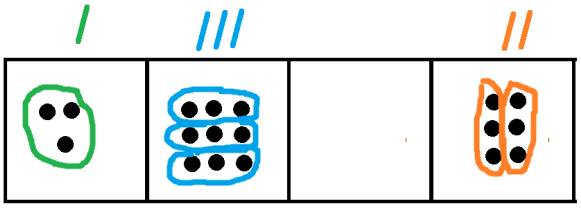
Let's start slowly with a division problem whose answer might be able to see right away.

What is $3906 \div 3$ *?*

The answer is 1302 .

If you think of 3906 as $3000 + 900 + 6$, then we can see that dividing by three then gives $1000 + 300 + 2$.

And we can really see this if we draw a picture of 3906 in a 1 ← 10 machine. We see groups of three: 1 group at the thousands level, 3 groups at the hundreds level, and 2 groups at the ones level.

3906 = 

That's it! We're doing division and seeing the answers division answers just pop right out!

Try doing $402 \div 3$ with just a dots-and-boxes picture. Do you see that unexplosions unlock this problem to reveal the answer 134 ?

And if you want to think deeply about what is really going on in these pictures (is it really this easy?) skip to the section "Deeper Explanation" in this chapter.

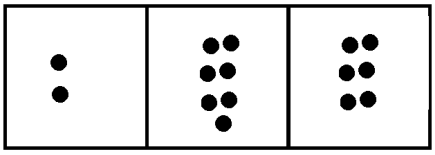
But if you are feeling ready to keep going right now … then let's keep going!

Long Division

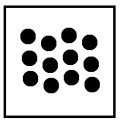
Division by single-digit numbers is all well and good. What about division by multi-digit numbers? People usually call that *long division*.

Let's consider the problem $276 \div 12$.

Here is a picture of 276 in a $1 \leftarrow 10$ machine.

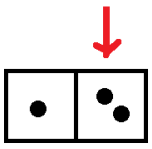
276 = 

And we are looking for groups of twelve in this picture of 276 . Here's what twelve looks like.

12 = 

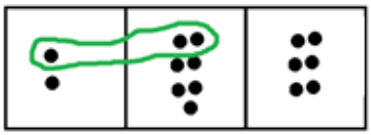
Actually, this is not right as there would be an explosion in our $1 \leftarrow 10$ machine. Twelve will look like one dot next to two dots. (But we need to always keep in mind that this really is a picture with all twelve

dots residing in the rightmost box.)

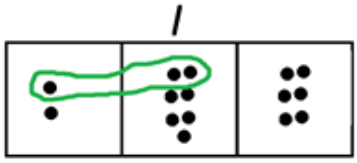
12 = 

Okay. So we're looking for groups of 12 in our picture of 276. Do we see any one-dot-next-to-two-dots in the diagram?

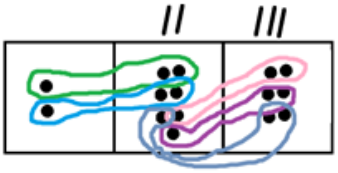
Yes. Here's one.

276 = 

Within each loop of 12 we find, the 12 dots actually reside in the right part of the loop. So we have found one group of 12 at the tens level.

276 = 

And there are more groups of twelve.

276 = 

We see a total of two groups of 12 at the tens level and three 12 s at the ones level. The answer to $276 \div 12$ is thus 23.

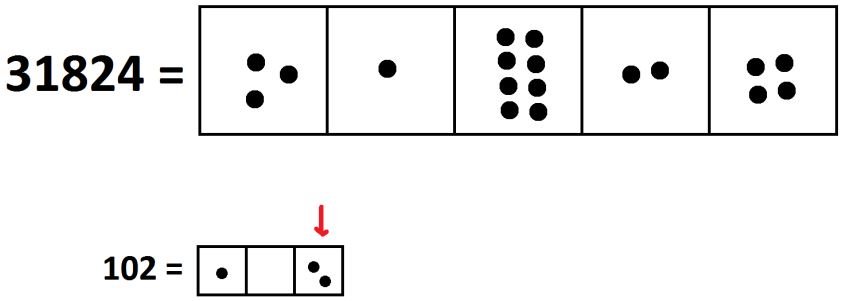
Here's are some practice questions you might, or might not, want to try. My answers to them appear at the end of this chapter.

1. Compute $2783 \div 23$ by the dots-and-boxes approach by hand.

2. Compute $3900 \div 12$ ..

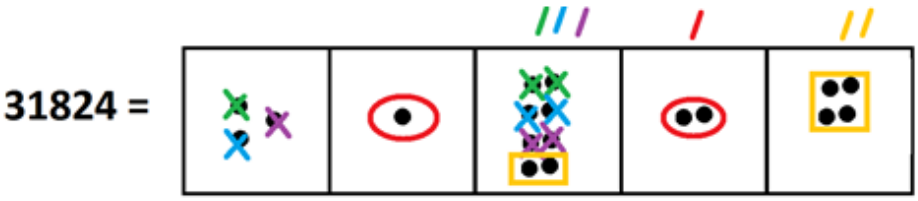
Let's do another example. Let's compute $31824 \div 102$.

Here's the picture.



Now we are looking for groups of one dot–no dots–two dots in our picture of 31824 . (And, remember, all 102 dots are physically sitting in the rightmost position of each set we identify.)

We can spot a number of these groups. (I now find drawing loops messy so I am drawing Xs and circles and boxes instead. Is that okay? Do you also see how I circled a double group in one hit at the very end?)



The answer 312 to $31824 \div 102$ is now apparent.

Here are some more questions to try, if you wish.

3. Compute $46632 \div 201$.

4. Show that $31533 \div 101$ equals 312 with a remainder of 21 .

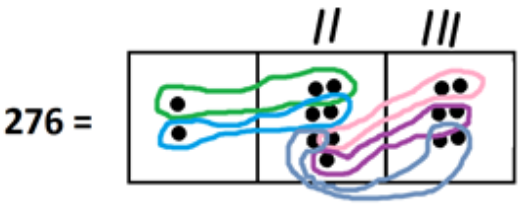
DIVISION BY TEN

Use dots-and-boxes to compute $2130 \div 10$. Can you explain why, with unexplosions, the answer will be 213 ? Do look for groups of ten in your picture.

Most people say that to divide a number that ends with zero by ten, just cross off the final zero from the number. Can you explain now why that action is sure to lead to the correct answer?

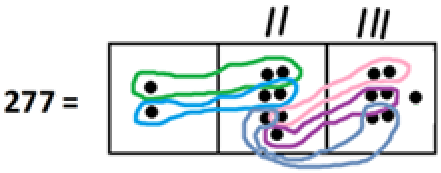
REMAINDERS

In the last section we saw that $276 \div 12$ equals 23.



Suppose we tried to compute $277 \div 12$ instead. What picture would we get? How should we interpret the picture?

Well, we'd see the same picture as before except for the appearance of one extra dot, which we fail to include in a group of twelve.



This shows that $277 \div 12$ equals 23 with a remainder of 1.

You might write this as

$$277 \div 12 = 23 \ R \ 1$$

or with some equivalent notation for remainders. (People use different notations for remainders in different countries.) Or you might be a bit more mathematically precise and say that $276 \div 12$ equals 23 with one more dot still to be divided by twelve:

$$277 \div 12 = 23 + \frac{1}{12}.$$

Here are some questions to try, if you want.

5. Compute $2789 \div 11$.
6. Compute $4366 \div 14$.
7. Compute $5481 \div 131$.

As you play with division in dots-and-boxes you might decide that it is actually good to always work from left to right in case there are remainders: we'd like all the "extra" dots we see appear in the lower places, the ones and tens places, rather than the higher thousands places, for instance. (But even if you don't choose to do this, you won't go wrong! Unexplosions will always be possible to help you out.)

DEEPER EXPLANATION

As one mulls on the long-division process you come to realize that there are subtle issues at play.

Let's take some time here to think through division more slowly. And let's start with the example whose answer we can write down right away.

What is $3906 \div 3$ *?*

Answer: 1302 .

What makes us able to see this answer so swiftly?

It seems natural to think of 3906 as $3000 + 900 + 6$. It is easy to divide each of these components by three.

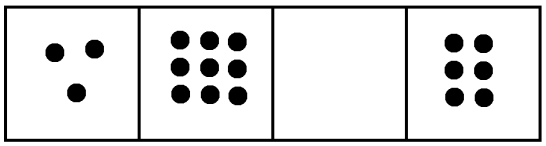
Dividing $3906 \quad = 3000 + 900 + 6$

by three gives

$$3906 \div 3 = 1000 + 300 + 2 = 1302 \ .$$

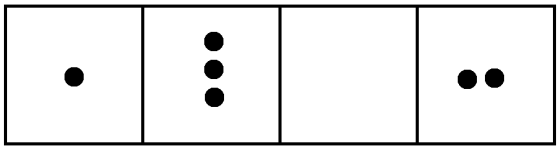
Great! And we see this natural decomposition too in a dots and boxes picture of 3906 . We literally see 3 thousands, 9 hundreds, and 6 ones.

3906 =



Dividing by three gives this picture.

1302 =



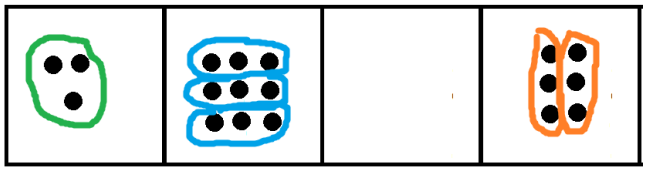
But let's probe even deeper into the workings of this final division step. What really happened here?

We can think of division as a task of grouping: $3906 \div 3$? is really asking

How many groups of three can you find among a collection of 3906 objects?

We know that there one thousand groups of three among 3000 dots, and three-hundred groups of three among 900 dots, and two groups of three among 6 dots. And our picture of 3906 actually shows this too.

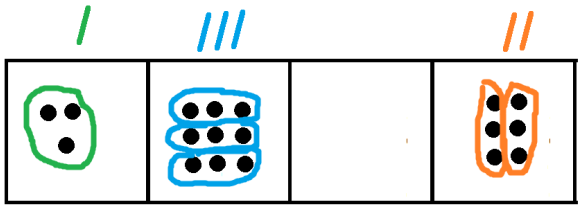
3906 =



If we did all the unexplosions, the green loop of dots would unexplode to give one-thousand green loops in the ones place. Each blue loop of dots unexplodes to make one-hundred blue loops in the ones place, and as there are three blue loops we get a total of three-hundred blue loops in the ones place. . We see that out picture is really one of one-thousand green loops, three-hundred blue loops, and two orange loops. We have 1302 groups of three.

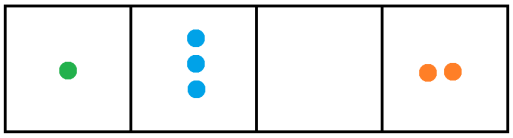
We can use tally marks to show that we have 1 group of three at the thousands level, 3 at the hundreds level, 0 at the tens level, and 2 at the ones level, again 1302 groups of three.

3906 =



And these tally marks show what happens if we were to actually divide by three: each group of three dots becomes one dot. We'd get this picture.

1302 =



This final picture shows how many groups of three we had in the original 3906. But we don't actually need to draw this final picture: the tally marks in the picture before it show this information too. So we

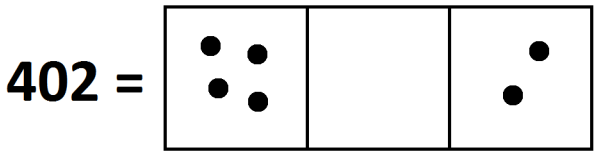
can stop drawing once we've figured out all the tallies.

Here's another practice question you might, or might not, want to do.

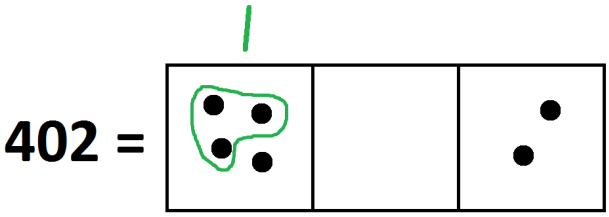
8. Draw a dots and boxes picture of the number 426 and use it explain why $426 \div 2$ equals 213. Try it on the app too.

Let's do another problem. Let's try $402 \div 3$.

Here's a picture of 402.

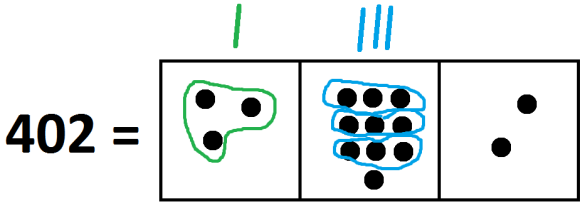


We're looking for groups of three in it. We see one group at the hundreds level. (This single loop really does represent one-hundred groups of three. Unexplosions give that.)

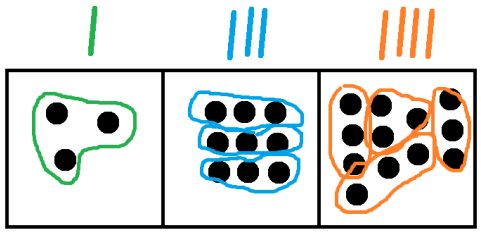


Now we seem to be stuck.

But an unexplosion gets us moving again!



And another unexplosion.

402 = 

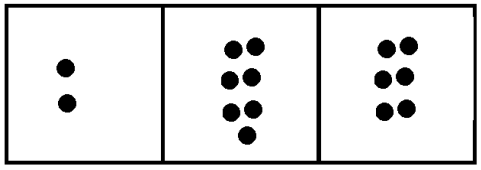
Now we see that there are one hundred, three tens, and four groups of three in the quantity 402 . Thus $402 \div 3 = 134$.

9. We just showed that $402 \div 3 = 134$. What do you think is the answer, then, to $404 \div 3$? What would you see in a picture of this division problem?

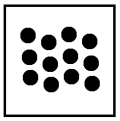
10. Compute $61230 \div 5$ by the dots-and-boxes approach.
(Does it get tiresome drawing dots? Do you have to actually draw them?)

Exactly the same thinking applies to multi-digit division too. We previously looked at $276 \div 12$.

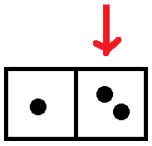
Here's picture of 276 .

276 = 

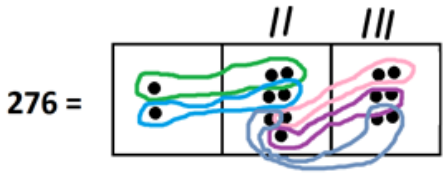
Here's what twelve dots look like.

12 = 

But in a $1 \leftarrow 10$ machine we'd really see them as one dot next to two dots after an explosion. (All twelve dots still really reside in the rightmost box.)

12 = 

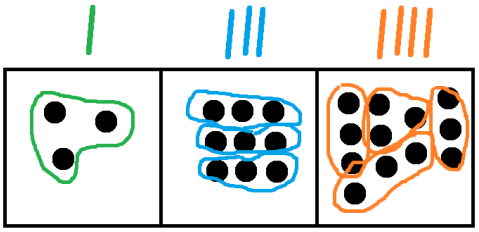
And when we hunt for groups of twelve in 276 , we get this picture.



We see two groups of 12 at the tens level and three 12 s at the ones level. The answer to $276 \div 12$ is indeed 23 .

THE TRADITIONAL ALGORITHM

Here's the dots and boxes way to show that $402 \div 3$ equals 134.

402 = 

This looks nothing like the approach one is usually taught in schools to solve division problems. For example, many schools have students compute $402 \div 3$ with an algorithm that looks something like this.

```
     134
3 ⟌402
   3↓
   10
    9
    12
    12
     0
```

At first glance this seems very mysterious, but it is really no different from the dots and boxes method. To see why, let's first explore an estimation method for division also often taught to students. It goes as follows:

To compute $402 \div 3$ we need to figure out the number of groups of three we can find in 402.

Let's first make a big guess, say, one hundred groups of three.

	Groups of 3
3 ⟌402	100

How much is left over after taking away one hundred groups of three? Answer: 102.

	Groups of 3
$3\overline{)402}$	100
$-\ 300$	
102	

How many groups of three are there in this remaining 102 ? Let's guess 30 .

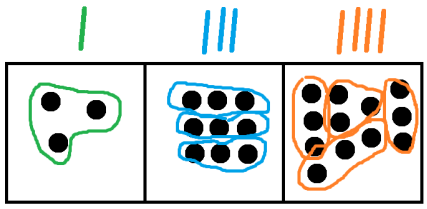
	Groups of 3
$3\overline{)402}$	100
$-\ 300$	
102	
$-\ 90$	30
12	

That leaves twelve. And there are four groups of three in twelve.

	Groups of 3
$3\overline{)402}$	100
$-\ 300$	
102	
$-\ 90$	30
12	
$-\ 12$	4
0	

This accounts for the entire count of 402 . We see that there are 134 groups of three in this count.

The dots and boxes method is doing exactly the same work, but purely visually.

402 = 

And the table we first presented is also identical to this estimation method. It was invented just to use

less ink as it doesn't write down quite as much. (It skips rewriting some digits.)

```
     134
3 ⟌402
   3↓
   10
    9↓
    12
    12
     0
```

```
                Groups of 3
3 ⟌402          100
  -300
   102
  -90           30
    12
  -12           4
     0
```

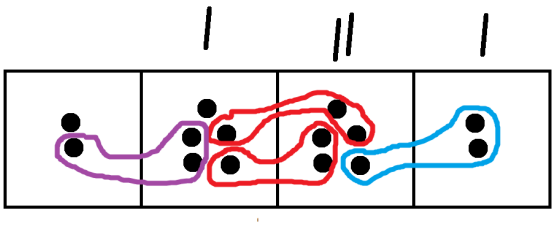
WILD EXPLORATION

Here is a "big question" investigation you might want to explore, or just think about. Have fun!

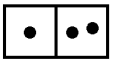
EXPLORATION: LEFT TO RIGHT? RIGHT TO LEFT? ANY ORDER?

When asked to compute $2552 \div 12$ Kaleb drew this picture, which he got from identifying groups of twelve working right to left.

2552 =



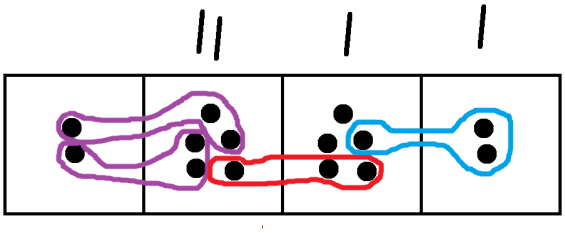
12 =



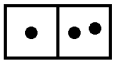
He said the answer to $2552 \div 12$ is 121 with a remainder of 1100.

Mabel, on the other hand, identified groups of twelve from left to right in her diagram for the problem.

2552 =



12 =



She concluded that $2552 \div 12$ is 211 with a remainder of 20.

Both Kaleb and Mabel are mathematically correct, but their teacher pointed out that most people would expect an answer with smaller remainders: both 1100 and 20 would likely be considered strange remainders for a problem about division by twelve. She also showed Kaleb and Mabel the answer to the problem that is printed in the textbook.

$$2552 \div 12 = 212 \ R \ 8$$

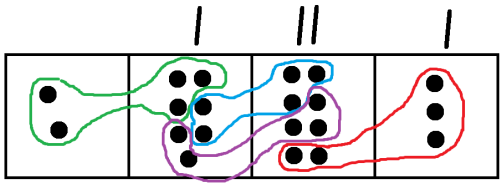
How could Kaleb and Mabel each continue work on their diagrams to have this textbook answer appear?

SOLUTIONS

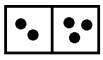
As promised, here are my solutions to the question posed.

1. $2783 \div 23 = 121$. (Can you see how I am getting efficient with me loop drawing?)

2783 =

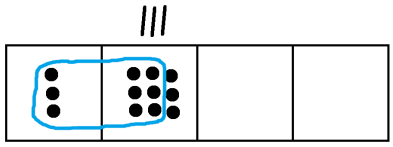


23 =



2. $3900 \div 12 = 325$. We need some unexplosions along the way.

3900 =



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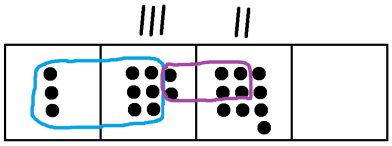

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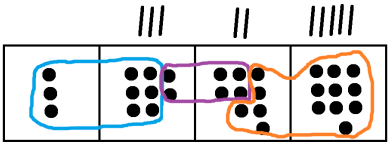
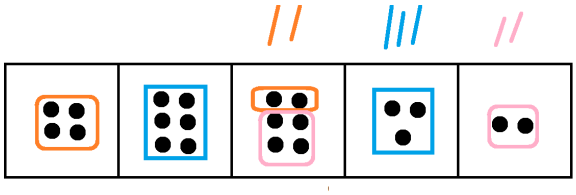


3. $46632 \div 201 = 232$.

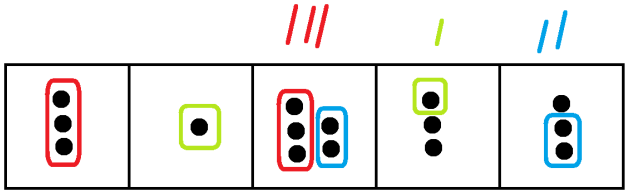
46632 =



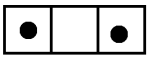
201 =

4.

31533 =

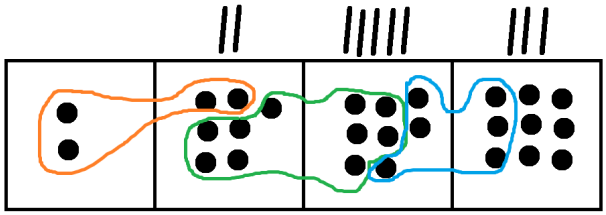


101 =

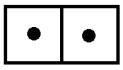


5. We have $2789 \div 11 = 253$ with a remainder of 6. That is, $2789 \div 11 = 253 + \frac{6}{11}$.

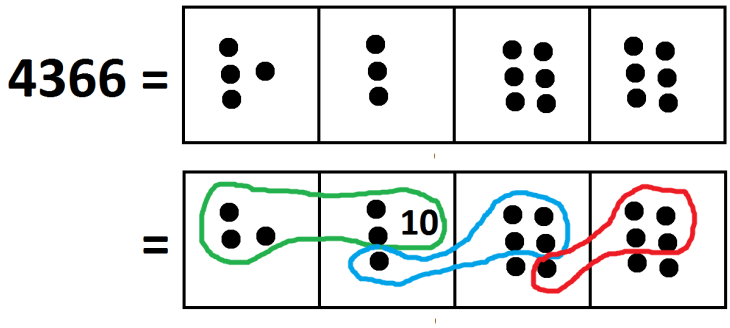
2789 =



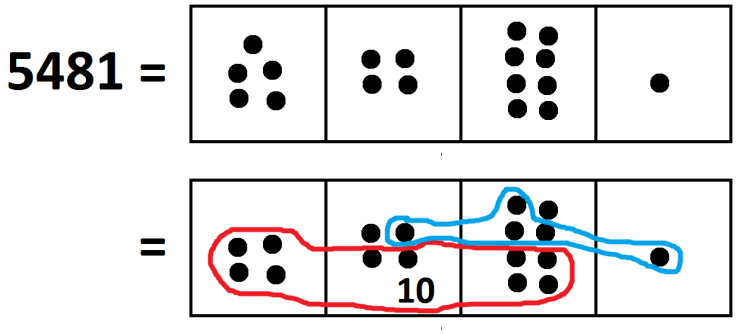
11 =



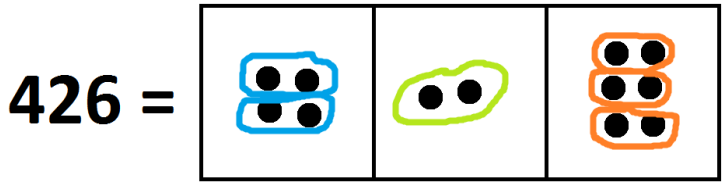
6. $4366 \div 14 = 311 + \frac{12}{14}$.



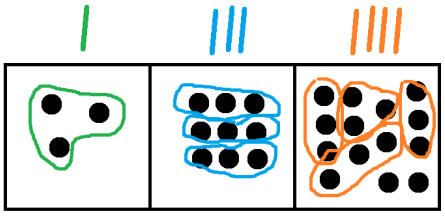
7. $5481 \div 131 = 41 + \frac{110}{131}$.



8. We see two groups of two at the hundreds level (all the dots in the blue loops would unexplode to make two-hundred blue sets of two at the ones level), one group of two at the tens level (all the dots in the green loop would unexplode to make ten green groups of two at the ones level), and three orange groups of two at the ones level. That makes for 213 groups of two.



9. In the picture for $404 \div 3$ we see two leftover dots unaccounted for.

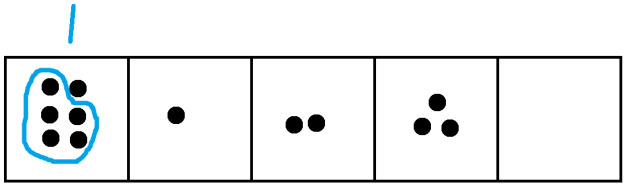
404 = 

So $404 \div 3$ equals 134 with a remainder of 2.

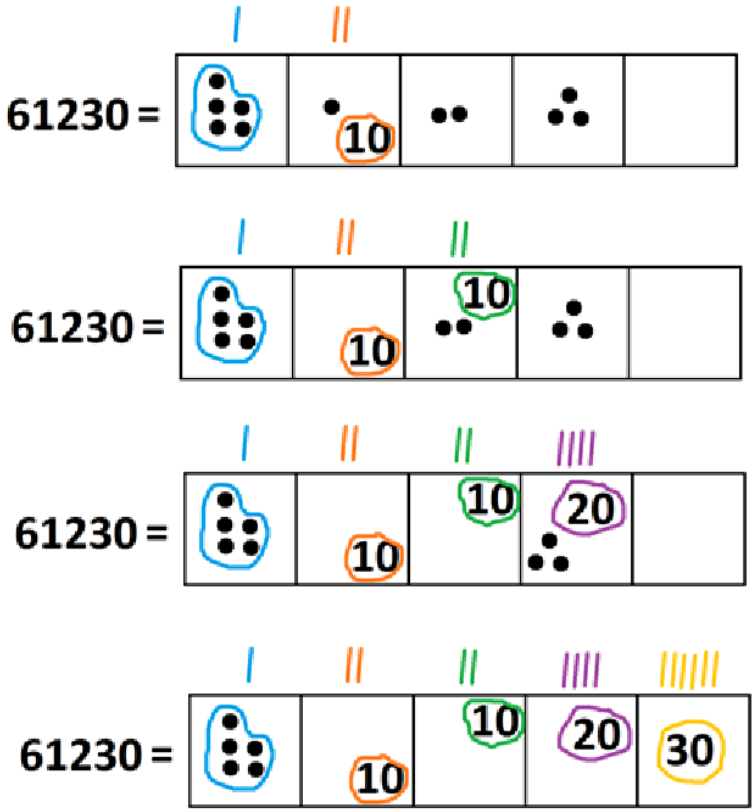
Note: We could regard this remainder as "two dots still to be divided by three," and so write

$$404 \div 3 = 134 + \frac{2}{3}.$$

10. We certainly see one group of five right away.

61230 = 

Let's perform some unexplosions. (And let's write numbers rather than draw lots of dots. Drawing dots gets tedious!)



We see $61230 \div 5 = 12246$.