OCR Further Pure 1

Complex Numbers

Section 1: Introduction to complex numbers

Notes and Examples

These notes contain subsections on

The number system

In your learning of mathematics, you have come across different types of number at different stages. Each time you were introduced to a new set of numbers, this allowed you to solve a wider range of problems.

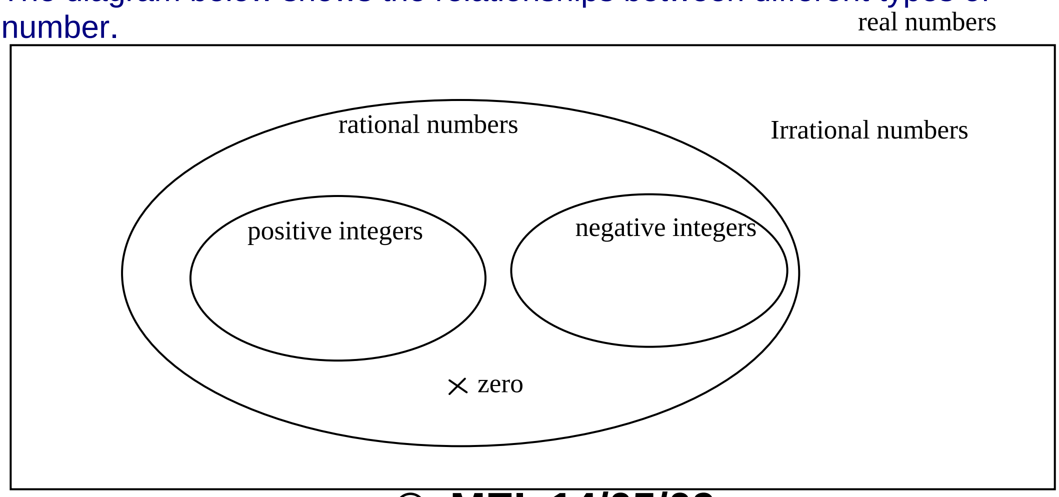
The first numbers that you came across were the counting numbers (natural numbers). These allowed you to solve equations like $x + 2 = 5$.

Later you would meet negative numbers, which allowed you to solve equations like $x + 5 = 2$, and rational numbers, which meant you could solve equations like $2x = 5$.

When irrational numbers were included, you could solve equations like $x^2 = 2$.

However, there are still equations which you cannot solve, such as $x^2 = -4$. You know that there are no real numbers which satisfy this equation. However, this equation, and others like it, can be solved using imaginary numbers, which are based on the number i, which is defined as $\sqrt{-1}$.

The diagram below shows the relationships between different types of number.



This type of diagram is called a Venn diagram (you may have met these before if you have studied any Statistics) and it shows the relationships between sets, in this case sets of numbers. This diagram deals with the real numbers, which include all numbers which you have come across until now. Notice that the positive and negative integers (whole numbers) are subsets of the rational numbers. This means that all integers are also rational numbers, but there are other rational numbers which are not integers, such as $\frac{3}{2}$ or $-\frac{7}{11}$. Similarly, all rational numbers are real numbers, but there are other real numbers which are not rational, such as $\sqrt{3}$ and π.

In this chapter you will see that the real numbers are also a subset of a larger set called the complex numbers. You will be looking at numbers which lie outside the set of real numbers. Complex numbers involve both real and imaginary numbers.

Adding and subtracting complex numbers

To add two complex numbers, you need to add the real parts and add the imaginary parts. Similarly, to subtract one complex number from another, deal with the real and imaginary parts separately.

Example 1
The complex numbers z and w are given by

$z = 3 + 2\text{i}$

$w = 1 - 4\text{i}$

Find:
(i) $z + w$ (ii) $z - w$ (iii) $w - z$

Solution

(i) $z + w = (3 + 2\text{i}) + (1 - 4\text{i})$
$= (3 + 1) + (2\text{i} - 4\text{i})$
$= 4 - 2\text{i}$

Add the real parts and add the imaginary parts

(ii) $z - w = (3 + 2\text{i}) - (1 - 4\text{i})$
$= (3 - 1) + (2\text{i} + 4\text{i})$
$= 2 + 6\text{i}$

Subtract the real parts and subtract the imaginary parts

(iii) $w - z = (1 - 4\text{i}) - (3 + 2\text{i})$
$= (1 - 3) + (-4\text{i} - 2\text{i})$
$= -2 - 6\text{i}$

For practice in examples like the one above, try the interactive questions ***Addition and subtraction of complex numbers***.

Multiplying complex numbers

Multiplication of two complex numbers is similar to multiplying out a pair of brackets. Each term in the first bracket must be multiplied by each term in the second bracket. You can then simplify, remembering that $i^2 = -1$.

Example 2
Find
(i) $(3 + 4i)(2 + i)$
(ii) $(2 - i)(3 + 2i)$
(iii) $(2 + 3i)(2 - 3i)$

Solution

(i) $(3 + 4i)(2 + i) = 6 + 3i + 8i + 4i^2$ (Multiply out the brackets)
$= 6 + 11i - 4$ (Using $i^2 = -1$)
$= 2 + 11i$

(ii) $(4 - i)(3 + 2i) = 12 + 8i - 3i - 2i^2$
$= 12 + 5i + 2$
$= 14 + 5i$

(iii) $(2 + 3i)(2 - 3i) = 4 - 6i + 6i - 9i^2$
$= 4 + 9$
$= 13$

For practice in examples like the one above, try the interactive questions ***Multiplication of complex numbers***.

Complex conjugates

In part (iii) of Example 2, the result of multiplying two complex numbers is a real number. This is always the case when the complex number $a + bi$ is multiplied by the complex number $a - bi$. The complex number $a - bi$ is called the **complex conjugate** of $a + bi$. For any complex number z, the complex conjugate is written as $\bar{z}$ or z^*.

For practice in complex conjugates, try the interactive questions ***Conjugate of a complex number***.

Division of complex numbers

The result that a complex number multiplied by its conjugate is always real is key to the technique of dividing complex numbers. By multiplying both the numerator and the denominator by the complex conjugate of the denominator, the denominator becomes real.

Example 3

Write $\dfrac{2+3i}{3-4i}$ in the form $a + bi$.

Solution

Multiply top and bottom by 3 + 4i (the complex conjugate of 3 – 4i)

$$\frac{2+3i}{3-4i} = \frac{(2+3i)(3+4i)}{(3-4i)(3+4i)}$$
$$= \frac{6+8i+9i-12}{9+16}$$
$$= \frac{-6+17i}{25}$$
$$= -\tfrac{6}{25} + \tfrac{17}{25}i$$

For practice in examples like the one above, try the interactive questions ***Dividing complex numbers***.

Equations with complex roots

When you first learned to solve quadratic equations using the quadratic formula, you found that some quadratic equation had no real solutions. However, using complex numbers you can find solve all quadratic equations.

Example 4

Solve the quadratic equation

$x^2 + 6x + 13 = 0$

Solution

Using the quadratic formula with $a = 1$, $b = 6$, $c = 13$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-6 \pm \sqrt{36 - 4\times1\times13}}{2\times1}$$
$$= \frac{-6 \pm \sqrt{-16}}{2}$$
$$= \frac{-6 \pm 4i}{2}$$
$$= -3 \pm 2i$$

The solutions of the equation are $x = -3 + 2i$ and $x = -3 - 2i$

Notice that the quadratic equation in Example 4 has two complex solutions which are a pair of complex conjugates. All quadratic equations with real coefficients have two solutions: either two real solutions (which could be a repeated solution) or two complex solutions which are a pair of complex conjugates.

The Flash resource ***Complex roots of quadratics*** shows how the complex roots of a quadratic equation are related to the graph of the quadratic function. You do not need to know this work for your exam, but it is interesting background work.

The next example shows how you can find a quadratic equation with roots at particular complex values. A quadratic equation with roots at $x = a$ and $x = b$ can be written as $(x - a)(x - b) = 0$, and this also applies to situations where the roots are complex numbers.

Example 5

Find the quadratic equation which has roots at $x = 4 + 2\text{i}$ and $x = 4 - 2\text{i}$.

Solution

$$(x - (4 + 2\text{i}))(x - (4 - 2\text{i})) = 0$$
$$(x - 4 - 2\text{i})(x - 4 + 2\text{i}) = 0$$
$$(x - 4)^2 - (2\text{i})^2 = 0$$
$$x^2 - 8x + 16 + 4 = 0$$
$$x^2 - 8x + 20 = 0$$

The two middle terms cancel out since this expression is of the form $(x - a)(x + a)$

The Flash resource ***Working with complex numbers*** tests you on multiplication, complex conjugates and equations with complex roots.

Equating real and imaginary parts

For two complex numbers to be equal, then the real parts must be equal and the imaginary parts must be equal. So one equation involving complex numbers can be written as two equations, one for the real parts, one for the imaginary parts.

The example below shows how this technique can be used to solve equations involving complex numbers. Two solutions are shown. In the first solution, the equation is treated in the same sort of way as for an equation involving real numbers, so division of complex numbers is used. In the second solution, the method of equating real and imaginary parts is used.

Example 6

Solve the equation

$$(3 - 2\text{i})(z - 1 + 4\text{i}) = 7 + 4\text{i}$$

Solution 1

$$(3 - 2\text{i})(z - 1 + 4\text{i}) = 7 + 4\text{i}$$

Divide both sides by $3 - 2\text{i}$

$$z - 1 + 4i = \frac{7+4i}{3-2i} = \frac{(7+4i)(3+2i)}{(3-2i)(3+2i)}$$

$$= \frac{21+14i+12i-8}{9+4}$$

$$= \frac{13+26i}{13} = 1+2i$$

Subtract -1 + 4i from each side

$z = 1 + 2i - (-1 + 4i)$

$= 2 - 2i$

Solution 2

Let $z = x + iy$

$(3 - 2i)(x + iy - 1 + 4i) = 7 + 4i$

$(3 - 2i)((x - 1) + i(y + 4)) = 7 + 4i$

$3(x - 1) - 2i(x - 1) + 3i(y + 4) - 2i^2(y + 4) = 7 + 4i$

$3(x - 1) - 2i(x - 1) + 3i(y + 4) + 2(y + 4) = 7 + 4i$

Equating real parts: $3(x - 1) + 2(y + 4) = 7 \Rightarrow 3x + 2y = 2$ ①

Equating imaginary parts: $-2(x - 1) + 3(y + 4) = 4 \Rightarrow -2x + 3y = -10$ ②

① × 2 $6x + 4y = 4$

② × 3 $-6x + 9y = -30$

Adding: $13y = -26$

$y = -2$

$x = 2$

$z = 2 - 2i$