MEI Further Pure Mathematics 1

Complex Numbers

Section 1: Introduction to complex numbers

Notes and Examples

These notes contain subsections on

The growth of the number system

Look at Figure 2.1. This type of diagram is called a Venn diagram (you may have met these before if you have studied any Statistics) and it shows the relationships between sets, in this case sets of numbers. This diagram deals with the real numbers, which include all numbers which you have come across until now. Notice that the positive and negative integers (whole numbers) are subsets of the rational numbers. This means that all integers are also rational numbers, but there are other rational numbers which are not integers, such as $\frac{3}{2}$ or $-\frac{7}{11}$. Similarly, all rational numbers are real numbers, but there are other real numbers which are not rational, such as $\sqrt{3}$ and π.

In this chapter you will see that the real numbers are also a subset of a larger set called the complex numbers. You will be looking at numbers which lie outside the set of real numbers.

Look at Activity 2.2. You may find it difficult to think of a problem which would lead to equation (v). One possible answer is the following:

You have two thermometers, A and B, one of which has been calibrated incorrectly, so that the reading on thermometer A is always 7 degrees below the reading on thermometer B. For which readings on thermometer A is the product of the two readings zero?

Don't spend too long thinking about (vi). Read on!

Quadratic equations with complex roots

When you first learned to solve quadratic equations using the quadratic formula, you found that some quadratic equation had no real solutions. However, using complex numbers you can find solve all quadratic equations.


Example 1
Solve the quadratic equation
$x^2 + 6x + 13 = 0$

Solution
Using the quadratic formula with $a = 1$, $b = 6$, $c = 13$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{36 - 4 \times 1 \times 13}}{2 \times 1} \\ &= \frac{-6 \pm \sqrt{-16}}{2} \\ &= \frac{-6 \pm 4\mathrm{i}}{2} \\ &= -3 \pm 2\mathrm{i} \end{aligned}$$

The solutions of the equation are $x = -3 + 2\mathrm{i}$ and $x = -3 - 2\mathrm{i}$

Notice that the quadratic equation in Example 1 has two complex solutions which are a pair of complex conjugates. All quadratic equations with real coefficients have two solutions: either two real solutions (which could be a repeated solution) or two complex solutions which are a pair of complex conjugates.

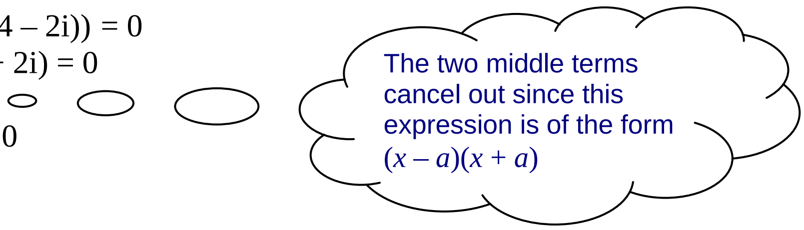
The Flash resource ***Complex numbers and the completed square form*** shows how the complex roots of a quadratic equation relate to the graph of the quadratic function. You do not need to know this work, but it is interesting extension work.

The next example shows how you can find a quadratic equation with roots at particular complex values. A quadratic equation with roots at $x = a$ and $x = b$ can be written as $(x - a)(x - b) = 0$, and this also applies to situations where the roots are complex numbers.


Example 2
Find the quadratic equation which has roots at $x = 4 + 2\mathrm{i}$ and $x = 4 - 2\mathrm{i}$.

Solution

$$\begin{aligned} &(x - (4 + 2\mathrm{i}))(x - (4 - 2\mathrm{i})) = 0 \\ &(x - 4 - 2\mathrm{i})(x - 4 + 2\mathrm{i}) = 0 \\ &(x - 4)^2 - (2\mathrm{i})^2 = 0 \\ &x^2 - 8x + 16 + 4 = 0 \\ &x^2 - 8x + 20 = 0 \end{aligned}$$



Working with complex numbers

In all calculations with complex numbers, remember to use the fact that $j^2 = -1$. When you multiply two complex numbers:

$$(x + yj)(u + vj)$$

you get one term (xu) involving just real numbers, two terms in j (xvj and yuj) and one term in j^2 (yvj^2) which is equal to $yv \times -1 = -yv$. So you end up with two real terms and two terms in j.

Example 3

Find

(i) $(2 + 3j)(4 + j)$

(ii) $(3 - 2j)(2 + 5j)$

Solution

(i) $(2 + 3j)(4 + j) = 8 + 2j + 12j + 3j^2$

$= 8 + 14j - 3$ ← using $j^2 = -1$

$= 5 + 14j$

(ii) $(3 - 2j)(2 + 5j) = 6 + 15j - 4j - 10j^2$

$= 6 + 11j + 10$ ← using $j^2 = -1$

$= 16 + 11j$

Make sure that you have grasped the terminology in this section, in particular that you know what is meant by the real part and the imaginary part of a complex number, and what is meant by a complex conjugate. Complex conjugates are very useful and you should take particular note of Activity 2.4.

You can practise calculations with complex numbers using the interactive questions ***Addition and subtraction of complex numbers***, ***Multiplying complex numbers*** and ***Complex conjugates***.

Equating real and imaginary parts

Examples 2.2, 2.3 and 2.4 demonstrate two very important techniques which are essential tools for working with complex numbers. The first, shown in Example 2.2, is known as "equating real and imaginary parts". This is based on the fact that for two complex numbers to be equal, then the real parts must be equal and the imaginary parts must be equal. So one equation involving complex numbers can be written as two equations, one for the real parts, one for the imaginary parts.

Example 4 below shows one application of equating real and imaginary parts. There are some problems of this type in Exercise 2B Question 2.

Example 4
Find real numbers a and b with $a > 0$ such that $(a + b\text{j})^2 = 16 - 30\text{j}$.

Solution

$$(a + b\text{j})^2 = 16 - 30\text{j}$$
$$a^2 + 2ab\text{j} + b^2\text{j}^2 = 16 - 30\text{j}$$
$$a^2 + 2ab\text{j} - b^2 = 16 - 30\text{j}$$

Equating imaginary parts: $2ab = -30 \Rightarrow b = -\frac{15}{a}$

Equating real parts: $a^2 - b^2 = 16$

Substituting: $a^2 - \frac{225}{a^2} = 16$

Multiplying through by a^2: $a^4 - 225 = 16a^2$

$$a^4 - 16a^2 - 225 = 0$$

This is a quadratic in a^2 and can be factorised:

$$(a^2 - 25)(a^2 + 9) = 0$$

Since a is real, $a^2 + 9$ cannot be equal to zero.
Since $a > 0$, the only possible solution is $a = 5$.

$$b = -\frac{15}{a} \Rightarrow b = -\frac{15}{5} = -3$$

$a = 5$, $b = -3$

Notice that in this example you are finding the square root of a complex number. The square root of 16 – 30j is 5 – 3j. You can find the other square root of this complex number by allowing a to be negative, so $a = -5$ and then $b = 3$. So the other square root is -5 + 3j. As with real numbers, one square root is the negative of the other.

Dividing complex numbers

The second important technique shown in this section is demonstrated in Examples 2.3 and 2.4. It is difficult to work with complex numbers in the denominator of an expression, so it can be very useful to be able to make the denominator real. This is sometimes known as “realising the denominator” of a fraction. The technique hinges on the fact that the product of a complex number and its conjugate is real (see Activity 2.4). So by multiplying both numerator and denominator of an expression by the conjugate of the denominator, you make the denominator real, without changing the value of the expression (*multiplying the top and bottom of a fraction by the same thing is equivalent to multiplying by 1, which leaves the value unchanged*).

You can practice this technique using the interactive questions ***Dividing complex numbers***.

Example 5 below shows how an equation involving complex numbers can be solved by using this technique of dividing complex numbers. There are some similar problems in Exercise 2B Question 4.

Example 5

Solve the equation

$$(3 - 2j)(z - 1 + 4j) = 7 + 4j$$

Solution

$$(3 - 2j)(z - 1 + 4j) = 7 + 4j$$

$$z - 1 + 4j = \frac{7+4j}{3-2j}$$

Divide both sides by 3 – 2j

$$= \frac{(7+4j)(3+2j)}{(3-2j)(3+2j)}$$

Multiply top and bottom by 3 + 2j
[this makes the denominator ('bottom') real]

$$= \frac{21+14j+12j-8}{9+4}$$

$$= \frac{13+26j}{13}$$

$$= 1 + 2j$$

$$z = 1 + 2j - (-1 + 4j)$$

Subtract -1 + 4j from each side

$$= 2 - 2j$$

Alternatively, you could solve this equation by equating real and imaginary parts. Put $z = x + y$j, multiply out and equate real and imaginary parts to get two equations in x and y. You can then solve these as simultaneous equations.

You can test yourself on the techniques in this chapter using the Flash program ***Working with complex numbers***.