# OCR Further Pure 1

## Complex Numbers

## Section 3: Modulus and argument

### Notes and Examples

These notes contain subsections on


You are familiar with describing a point in the plane using Cartesian coordinates. However, this is not the only way of describing the location of a point. One alternative is to give its distance from a fixed point (usually the origin) and a direction (in this case the angle between the line connecting the point to the origin, and the positive real axis).

This is a common method of describing locations in real life: you might say that a town is "50 miles north-west of London", or when walking in open countryside your map might show you that you need to walk 2 miles on a bearing of 124°.

In mathematics there are some situations in which this method of describing points is more convenient than Cartesian coordinates.

### The modulus of a complex number

The modulus of a complex number $z$ is the distance of the point representing $z$ from the origin on the Argand diagram. Notice that this definition also holds for real numbers on the number line: the modulus (or absolute value) of a real number is its distance on the number line from zero.

In the same way, $|z - w|$ (or $|w - z|$) is the distance of the point representing $z$ from the point representing $w$. This also holds for real numbers on the number line: the distance of a real number $x$ from a real number $y$ on the number line is $|x - y|$ (or $|y - x|$). For example, the distance between 2 and -3 on the number line is $|2 - (-3)| = 5$.

**Example 1**
Given that $z = 2 + 5\text{i}$ and $w = 3 - \text{i}$, find
(i) $|z|$
(ii) $|w|$
(iii) $|z - w|$

**Solution**

(i) $|z| = \sqrt{2^2 + 5^2} = \sqrt{29}$

(ii) $|w| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$

(iii) $z - w = 2 + 5i - (3 - i) = -1 + 6i$

$|z - w| = \sqrt{(-1)^2 + 6^2} = \sqrt{37}$

## Sets of points on the Argand diagram involving the modulus

Try using the Flash resource ***Investigation of loci*** to investigate loci of the form $|z - a| = r$. Move the point $z$ around and find different positions for $z$ for which the value of $|z - a|$ is equal to the value of $r$. What sort of shape do these positions form? Click the "eye" to check.
There is also an ***interactive spreadsheet*** which investigates this type of locus.

This type of locus can be summed up as follows:

- Read $|z - (a + bi)|$ as 'the distance from $z$ to the point $a + bi$
  [note $|z - a + bi|$ can be written $|z - (a - bi)|$, which is the distance from $z$ to the point $(a - bi)$].
- All sets of points given by $|z - (a + bi)| = r$ can be represented by a circle, centre $a + bi$, radius $r$.
- $|z - (a + bi)| \leq r$ represents the circle and its interior.
- $|z - (a + bi)| < r$ represents the interior of the circle (but not the circle itself). In this case you should draw the circle as a dotted line, to show that it is not included in the set of points.
- $|z - (a + bi)| \geq r$ represents the circle and its exterior.
- $|z - (a + bi)| > r$ represents the exterior of the circle (but not the circle itself). Again, you should draw the circle as a dotted line, to show that it is not included in the set of points.

Now use the Flash resource again to look at sets of points given by $|z - a| = |z - b|$. Move the point $z$ around and find different positions for which $|z - a| = |z - b|$. What sort of shape do these positions form? Click the "eye" to check.

You should be able to see that $|z-(a+bi)| = |z-(c+di)|$ means that $z$ is the same distance from the points representing the complex numbers $a + bi$ and $c + di$. The set of points for which this is true is the perpendicular bisector of a line connecting the points $a + bi$ and $c + di$.

$|z-(a+bi)| \leq |z-(c+di)|$ means that $z$ is nearer to $a + bi$ than to $c + di$, so this represents the side containing the point a + bi of the perpendicular bisector of the line joining $a + bi$ to $c + di$. As with the circles, in the case of $\leq$ or $\geq$ the perpendicular bisector itself is included, so is shown in as a solid line, but for $<$ or $>$ the line must be shown as dotted as it is not included.

# OCR FP1 Complex nos Section 3 Notes and Examples

## The argument of a complex number

Finding the argument of a complex number involves using some knowledge of Trigonometry from C2, including radians, and angles greater than 90°. You also need to know the values of the sine, cosine and tangent for common angles such as 30° $\left(\frac{\pi}{6}\text{ radians}\right)$, 45° $\left(\frac{\pi}{4}\text{ radians}\right)$ and 60° $\left(\frac{\pi}{3}\text{ radians}\right)$.
However, if you haven't covered this work yet, don't worry. Look at the additional notes which give some help on these topics. (If you have already covered the work on Matrices and Transformations, the work you did on rotation matrices will have given you some confidence in dealing with angles greater than 90°, though you still may need some help with radians.)

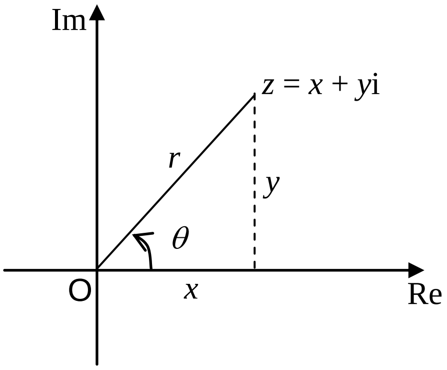
You can use the Flash resource ***The Argand diagram*** to investigate the modulus and argument of complex numbers.

The diagram shows that the argument $\theta$ of the complex number $z = x + y\text{i}$ satisfies the equations

$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$$

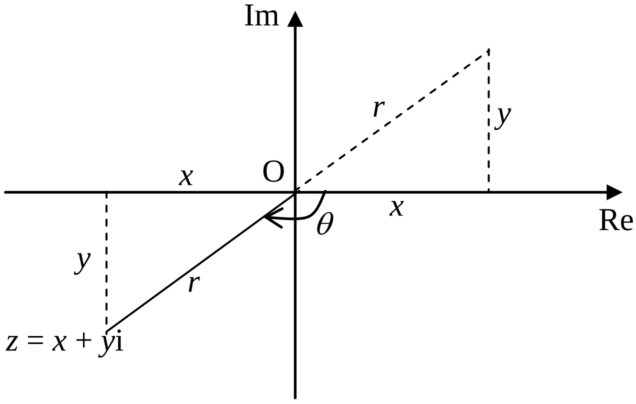
$$\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\tan\theta = \frac{y}{x}$$



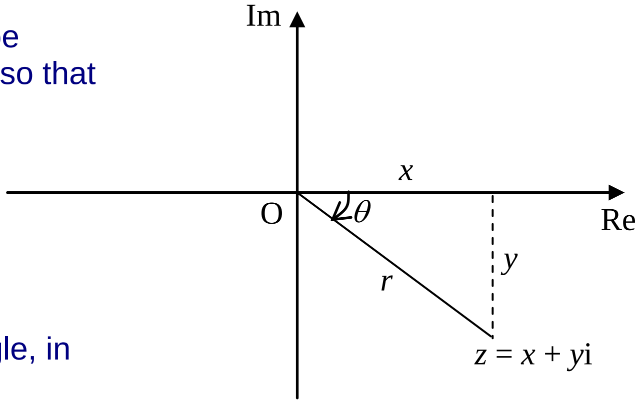

In the diagram, the complex number $z$ lies in the first quadrant, since both $x$ and $y$ are positive. So tan $\theta$ is positive, and to find the value of $\theta$, you just need to find $\arctan\frac{y}{x}$.

However, there is another possibility for which tan $\theta$ is positive. If both the real part and the imaginary part of $z$ are negative, $-\pi < \theta \leq -\frac{\pi}{2}$. In this case $z$ is in the third quadrant.
However, as tan $\theta$ is positive, using your calculator to find $\arctan\frac{y}{x}$ will give you the corresponding angle in the first quadrant, (see diagram) where $x$ and $y$ are both positive. To find the correct argument, you need to subtract $\pi$.



Next we need to look at the cases where $\tan\theta$ is negative.

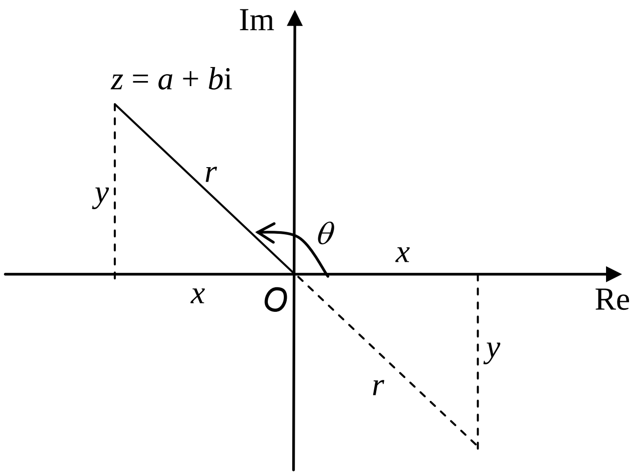
One possibility is for the real part of $z$ to be positive and the imaginary part negative, so that $-\frac{\pi}{2} \leq \theta < 0$. In this case, $z$ is in the fourth quadrant.




To find the value of $\theta$, find $\arctan\frac{y}{x}$. Your calculator should give you a negative angle, in the fourth quadrant as required.

The value of $\tan\theta$ is also negative if the real part of $z$ is negative, but the imaginary part is positive, so that $\frac{\pi}{2} \leq \theta \leq \pi$. In this case $z$ is in the second quadrant.

Using your calculator to find $\arctan\frac{y}{x}$ will give you the corresponding angle in the fourth quadrant, with $x$ positive and $y$ negative (see diagram). To find the correct angle in the second quadrant, you need to add $\pi$.




**Example 2**
Find the modulus and argument of each of the following complex numbers.
(i) $z = 4 + 3\text{i}$ (ii) $z = -1 + \text{i}$ (iii) $z = -1 - \sqrt{3}\text{i}$

**Solution**
(i) $z = 4 + 3\text{i}$
$|z| = \sqrt{(3^2 + 4^2)} = 5$
Since $z$ lies in the first quadrant, $\arg z = \arctan\frac{3}{4} \approx 0.644$

(ii) $z = -1 + \text{i}$
$|z| = \sqrt{(1^2 + 1^2)} = \sqrt{2}$
Since $z$ lies in the second quadrant, $\arg z = \arctan(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

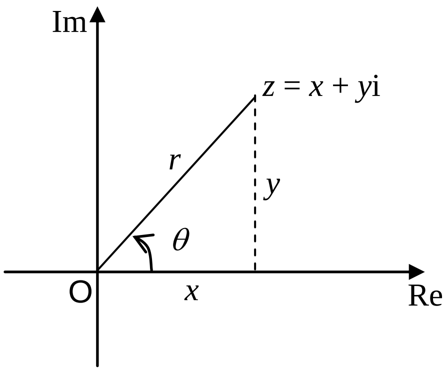
(iii) $z = -1 - \sqrt{3}\text{i}$
Modulus = $\sqrt{1+3} = 2$

Since $z$ lies in the fourth quadrant, $\arg z = \arctan\sqrt{3} - \pi = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

Sometimes you may want to find a complex number if you are given its modulus and argument.

You can see from the diagram that

$x = r\cos\theta$
$y = r\sin\theta$




These relationships allow you to find the real and imaginary parts of a complex number with a given modulus and argument.

**Example 3**

Find the complex numbers with the given modulus and argument, in the form $x + \mathrm{i}y$.

(i) $|z| = 3, \quad \arg z = \frac{3\pi}{4}$

(ii) $|z| = 2, \quad \arg z = -\frac{\pi}{6}$

**Solution**

(i) $r = 3,\ \theta = \frac{3\pi}{4}$

$$x = 3\cos\frac{3\pi}{4} = 3 \times -\frac{1}{\sqrt{2}} = -\frac{3}{\sqrt{2}}$$

$$y = 3\sin\frac{3\pi}{4} = 3 \times \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$z = -\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}\mathrm{i}$$

(ii) $r = 2,\ \theta = -\frac{\pi}{6}$

$$x = 2\cos\left(-\frac{\pi}{6}\right) = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$y = 2\sin\left(-\frac{\pi}{6}\right) = 2 \times -\frac{1}{2} = -1$$

$$z = \sqrt{3} - \mathrm{i}$$

# OCR FP1 Complex nos Section 3 Notes and Examples

You can test yourself on this work using the interactive questions ***Complex numbers: polar form***.

## Sets of points on the Argand diagram involving the argument

All sets of points of the form $\arg(z-(a+b\text{i}))=\theta$ consist of a half-line from the point $a + b\text{i}$ in the direction $\theta$.

Sets of points of the form $\theta_1 \le \arg(z-(a+b\text{i})) \le \theta_2$ consist of the two half-lines $\arg(z-(a+b\text{i}))=\theta_1$ and $\arg(z-(a+b\text{i}))=\theta_2$ and the region between them.

Be careful with sets of points of the form $\arg(z-(a+b\text{i})) \le \theta$ or $\arg(z-(a+b\text{i})) \ge \theta$. Make sure that you know where the region starts and ends. Remember that the possible values of $\arg z$ are given by $-\pi < \arg z \le \pi$, and that any line which is not included in the set of points should be shown as dotted (see Example 4 below).

**Example 4**

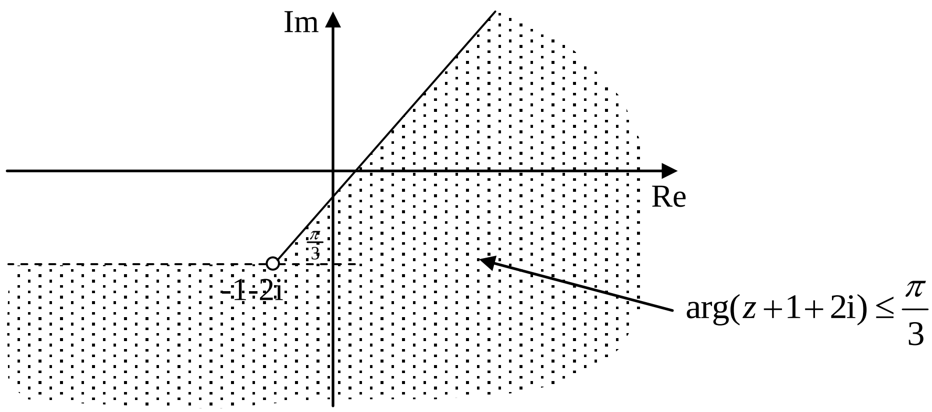
Draw Argand diagrams to show each of the following sets of points.

(i) $\arg(z+1+2\text{i}) \le \dfrac{\pi}{3}$

(ii) $\arg(z-2-\text{i}) > \dfrac{3\pi}{4}$

**Solution**

(i) $\arg(z+1+2\text{i}) \le \dfrac{\pi}{3}$ means that $\arg(z+1+2\text{i})$ can take any value between $-\pi$ and $\dfrac{\pi}{3}$, including $\dfrac{\pi}{3}$ but not $-\pi$. The half-line from $-1-2\text{i}$ in the direction $-\pi$ is therefore shown dotted, and the half-line from $-1-2\text{i}$ in the direction $\dfrac{\pi}{3}$ is shown as solid. The point $-1-2\text{i}$ is not included in the region, since $\arg(z+1+2\text{i})$ is not defined where $z=-1-2\text{i}$, so this point is shown by an open circle.



(ii) $\arg(z-2-\text{i}) > \frac{3\pi}{4}$ means that $\arg(z-2-\text{i})$ can take any value between $\frac{3\pi}{4}$ and $\pi$, including $\pi$ but not $\frac{3\pi}{4}$ since the inequality involves > rather than ≥. The half-line from 2 + i in the direction $\frac{3\pi}{4}$ is therefore shown dotted, and the half-line from 2 + i in the direction $\pi$ is shown solid. The point 2 + i is not included in the region, since $\arg(z-2-\text{i})$ is not defined where $z$ = 2 + i, so this point is shown by an open circle.