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TABLE SQUARE SUMS

1	2	3	4	5	6	7	8
2	4	6	8	10	12	14	16
3	6	9	12	15	18	21	24
4	8	12	16	20	24	28	32
5	10	15	20	25	30	35	40
6	12	18	24	30	36	42	48
7	14	21	28	35	42	49	56
8	16	24	32	40	48	56	64

What is the sum of the numbers in the top row?

… the second row?

… the fourth row?

… the bottom row?

What is the sum of all the numbers in the square?

What about a 10 by 10 square or an *n* by *n* square?

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Teacher Notes:

The key to success here is the realisation that the sum of each row is a multiple of the sum of the top row. The sum of the top row is 36, the eighth triangle number, which can be calculated as a half of 8 × 9. Hence, the sum of all the eight rows is:

$$36(1+2+3+4+5+6+7+8) = 36^2 = 1296$$

It follows that the sum of the numbers in a multiplication table square is the square of the sum of the top row which is equivalent to the square of the corresponding triangle number.

So, in the case of an *n* by *n* table square the sum of all the numbers is given by:

$$\left(\frac{1}{2}n(n+1)\right)^2 = \frac{1}{4}n^2(n+1)^2$$

This can be written as $\left(\sum_{i=1}^{n} i\right)^2$, which, interestingly, is the formula for the sum of the cubes $\sum_{i=1}^{n} i^3$.

Thus the given table square must be the sum of the first eight cubes. Find the sum of all the numbers in the fifth column and the fifth row up to 25, which they have in common, and that will help you to see why.

$$5+10+15+20+25+20+15+10+5 = 125 = 5^3$$

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ONE LESS THAN A SQUARE

$1^2 - 1 = \mathbf{0}$

$2^2 - 1 = \mathbf{3}$

$3^2 - 1 = 8$

$4^2 - 1 = \mathbf{15}$

$5^2 - 1 = \mathbf{24}$

$6^2 - 1 = 35$

$7^2 - 1 = \mathbf{48}$

$8^2 - 1 = \mathbf{63}$

Alternate numbers are multiples of 8. Can you explain why?

The numbers shown in bold are multiples of 3. Can you explain why?

Try expressing each as a product of factors.

Look at it algebraically.

Teacher Notes:

Since the squares alternate between odd and even the sequence does the same, but that is not sufficient to explain why alternate numbers are multiples of eight. Similarly every third number cannot be a multiple of three, but that does not explain by all the intervening numbers are.

$$1^2 - 1 = \mathbf{0} = 0 \times 2$$
$$2^2 - 1 = \mathbf{3} = 1 \times 2$$
$$3^2 - 1 = 8 = 2 \times 2$$
$$4^2 - 1 = \mathbf{15} = 3 \times 2$$
$$5^2 - 1 = \mathbf{24} = 4 \times 2$$
$$6^2 - 1 = 35 = 5 \times 2$$
$$7^2 - 1 = \mathbf{48} = 6 \times 2$$
$$8^2 - 1 = \mathbf{63} = 7 \times 2$$

With each number expressed as a product of two factors, the even numbers are products of consecutive even numbers, one of which is therefore a multiple of 4. So, the product is a multiple of eight. Algebraically the odd square numbers are of the form $(2n - 1)^2$. Subtracting one and simplifying gives $4n(n - 1)$, which is a multiple of eight, because one of the two consecutive numbers $n - 1$ and n must be even.

The cases where we have multiples of three have squares of the form $(3n - 1)^2$ or $(3n + 1)^2$. Subtracting one gives $9n^2 - 6n$ or $9n^2 + 6n$ — clearly a multiple of three in each case.

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A PUZZLE WITH TWO DICE

Give two dice – say blue and yellow – to a student and ask them to throw them and then:

a) multiply the blue score by 5
b) add 7 to the answer
c) double the result
d) add on the yellow score
e) tell you the total.

You can instantly tell them the two scores by subtracting 14 from the total. The tens digit is the blue score and the units digit is the yellow score.

Can you explain why this works?

Repeating the steps algebraically with b for the blue score and y for the yellow score gives:

a) $5b$
b) $5b + 7$
c) $10b + 14$
d) $10b + 14 + y$
e) subtracting 14 gives $10b + y$.

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Teacher Notes:

Problems like this provide a useful motivation for using some simple algebra both to show why they work and to devise similar problems of the same kind. The procedure could be modified, for example, by adding a number other than 7 and by doubling before multiplying by 5.

Devising a similar procedure for three dice provides an interesting challenge, If we denote the scores on a blue, yellow and green dice by b, y and g, then we need to end up with an expression of the form 1(X)b +10y g plus a three digit number that we can subtract. Here is one way to do this.

Multiply the blue score by 10	$10b$
Add 9	$10b + 9$
Add the yellow score	$10b + y + 9$
Double	$20b + 2y + 18$
Add 7	$20b + 2y + 25$
Multiply by 5	$100b + 10y + 125$
Add the green score	$100b + 10y + g + 125$

The original scores are the three digits of the number obtained by subtracting 125 from the result.

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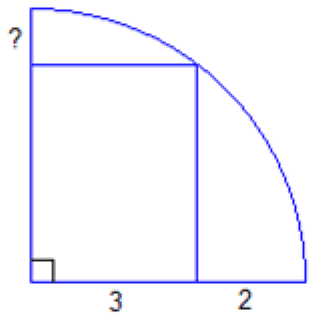
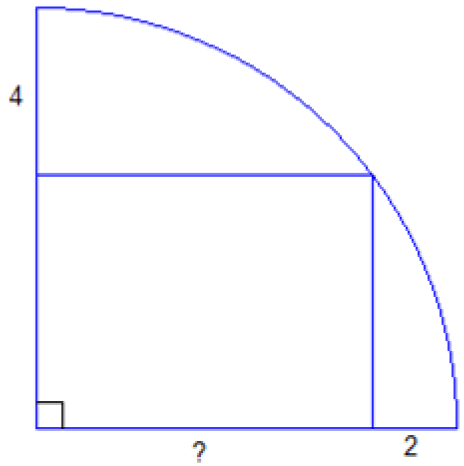
FIND THE LENGTHS

Find the missing lengths in these two diagrams.





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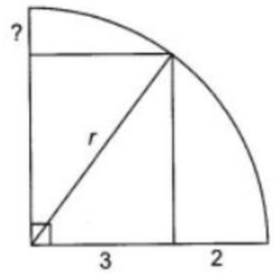
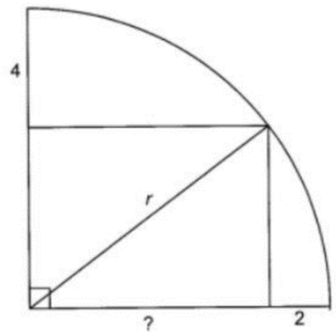
Teacher Notes:

In each case the key is to realise that the diagonal of the rectangle is the radius of the circle.



Since $r = 3 + 2 = 5$ the right angled triangles are $3, 4, 5$.

The unknown length is therefore $5 - 4 = 1$



Pythagoras' theorem gives a quadratic equation:

$$(r - 2)^2 + (r - 4)^2 = r^2$$

$$r^2 - 12r + 20 = 0 \Rightarrow (r - 2)(r - 10) = 0 \Rightarrow r = 2 \text{ or } 10$$

Since 10 is the only valid solution for the radius, the unknown length is 8.

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FOUR FIBONACCI NUMBERS

Look at four consecutive numbers taken from the Fibonacci Sequence, for example:

3 5 8 13

Notice that:

$$13 + 3 = 2 \times 8$$

The sum of the first and fourth terms is double the third term.

$$13 - 3 = 10 = 2 \times 5$$

The difference between the first and fourth terms is double the second term.

$$8^2 - 5^2 = 64 - 25 = 39 = 3 \times 13$$

The difference between the squares of the second and third terms is equal to the product of the first and fourth terms.

Are these results generally true and, if so, can you prove them?

Teacher Notes:

Four consecutive terms from the Fibonacci sequence can be denoted by a, b, a and $a + 2b$.

The sum of the first and fourth terms is $2a + 2b$, which is double the third term.

The difference between the first and fourth terms is $2b$, which is double the second term.

The difference of the squares of the second and third terms is $(a+b)^2 - b^2 = a(a+2b)$, which is the product of the first and fourth terms.

So, all three results are true for any set of four consecutive terms in the Fibonacci sequence. Indeed they are true for any other sequence where each term is obtained by adding the two previous terms. An example is provided by the Lucas sequence:

1 3 4 7 11 18 29

Another property, linked to the sum and difference discussed above, holds for these sequences: the difference of the squares of the first and fourth terms is four times the product of the second and third terms. This is shown by the identity:

$$(a+2b)^2 - a^2 = 2b(2a+2b) = 4b(a+b)$$

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A PAIR OF TWO DIGIT NUMBERS

With four consecutive digits make a pair of two digit numbers and add them.

For example, using 2, 3, 4, 5:

$$35 + 42 = 77$$

$$54 + 32 = 86$$

$$45 + 23 = 68$$

Can you find any more examples with the same four digits?

What do you notice?

Can you explain what is happening?

What happens with different sets of four consecutive digits?

What happens with a random set of digits?

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Teacher Notes:

The four digits $2, 3, 4$ and 5 can be arranged in $4! = 24$ different ways, but the same total is produced by four different arrangements by swapping the tens digits or the units digits. Thus it seems that there are six different totals.

Alternatively we can argue that with the four digits $2, 3, 4$ and 5 there are six ways of choosing two digits. If these are taken in turn as the tens digits, that again seems to give six possible totals, because it does not make any difference to the total which way round the two remaining digits are placed in the units positions. By swapping the units or tens digits as before, there are four arrangements that give each total, confirming that the total is 24.

$24 + 35 = 59$	$23 + 45 = 68$
$23 + 54 = 77$	$32 + 45 = 77$
$32 + 54 = 86$	$42 + 53 = 95$

Using the procedure for listing the six different totals as shown above, we find that one total is repeated, so in fact there are only five different totals, but one of them can be obtained in eight different ways. That arises because $2 + 5 = 3 + 4$, which gives us the special case of 77 where the two digits are the same.

The same results will apply to any set of four different digits apart from the property that gave rise to the 77s above. That only arises in the special cases where there are two pairs of numbers which have the same sum.

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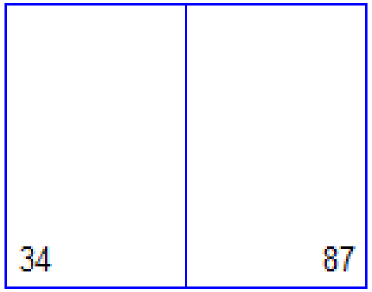
HOW MANY PAGES?



The middle section of a magazine has been removed to reveal pages 34 and 87.

- Which page is opposite page 36?
- Which page is opposite page 89?
- Which page is opposite page 54?
- What are the two middle pages?
- How many pages are there altogether?

Teacher Notes:

An obvious thing to do in the classroom is to split a newspaper or magazine into double page spreads and give one to each pair of students with some suitably adapted questions.

Where do we begin? Some will spot the general relationship between the page numbers quickly, but others will need to think in an experimental way before the result emerges. For example, we can count on to page 36 and then realise that for the other half of the two page sheet we will be counting back from 87 to 85. So, 36 goes with 85. By counting on in the same way 89 goes with 32.

54 is paired up with 67: the two page numbers are obtained by adding 20 to 34 and subtracting 20 from 87. The two middle pages can be found by continuing this addition and subtraction process until two consecutive page numbers are arrived at.

Finding the total number of pages may at first sight look like a different type of problem, until you realise that you need the page that goes with page 1 which involves adding and subtracting 33 from the given page numbers. So there are 120 pages.

Of course, it is all very simple once you realise that the sum of these opposite page numbers is constant, but this is something for students to discover for themselves, not something to tell them. It is very satisfying to establish such a neat way of determining the total number of pages from one pair of page numbers.