A NUMBER AND ITS SQUARE

Add a number to its square.

Try it with several numbers.

How can you prove that the sum is always even?

Proof 1

The square of an even number is even:

EVEN + EVEN = EVEN.

The square of an odd number is odd:

ODD + ODD = EVEN.

Proof 2

Let the number be n .

Since $n^2 + n = n(n+1)$, the sum is the product of two consecutive numbers which must be even.

It becomes obvious after adding a few numbers to their squares that the result is always even. Students should be challenged to find an explanation of this for themselves and then to share their thoughts with others. The two most likely explanations are the two described here.

The first proof is the simplest and does not require any algebra if we accept that the squares of even numbers are even and those of odd numbers are odd and that adding two evens or two odds always gives an even number.

For most students these results can be taken as intuitively obvious, but for some it may be interesting at some stage to justify the statements algebraically by referring to these identities

$$(2n^2\ ) = 4n^2 \text{ and } (2n-1)^2 = 4n^2 - 4n + 1,$$

$$2m + 2n = 2(m+n) \text{ and } (2m-1) + (2n-1) = 2(m+n-1)\ .$$

The second proof provides a completely different approach by expressing $n^2 + n$ as a product of two factors. Since with any pair of consecutive numbers one must be even, their product must also be even.

Another nice example which uses this idea of a product of factors is to show that the difference between a number and its cube is a multiple of 6.

$$n^3 - n = n(n^2 - 1) = (n-1)n(n+1)\ .$$

Writing the product in this way makes it clear that it is a product of three consecutive numbers. Since one must be even and one a multiple of three, the product is a multiple of six.

TWELVE CUBED

$12^3 = 1728$

Use this result to calculate:

120^3	$1728 \div 12$
1.2^3	$1728 \div 12^2$
$1.2 \times 12 \times 120$	$1728 \div 6$
0.12^3	$1728 \div 6^2$
6^3	$1728 \div 24$
$6^3 \times 2^3$	$1728 \div 24^2$
$12^3 \div 6^3$	$1728 \div 48$
$12^3 \div 6^2$	$1728 \div 48^2$

What can you work out from $12^3 = 1728$?

Working on examples like these reinforces arithmetical understanding and mental calculation skills. Their linked nature is in many respects more useful than a typical text book exercise, where each question is commonly independent of the others. Students should be encouraged to discuss and compare the methods that they use. Particular examples which illustrate key points can be used for whole class discussion.

Every question provides a wealth of possibilities which reinforce connections. Here are some thoughts on two of the examples.

With $1.2 \times 12 \times 120 = 1728$ it is necessary to decide where to place the decimal point in the answer. Students often have difficulty applying the rule about counting the number of digits after the point in such an example. The presence of 120 means we require a decimal point in 17280 with one digit after the point, giving 1728 from 1728.0. It is perhaps simpler to estimate: since $1 \times 10 \times 100 = 1000$ and $2 \times 20 \times 200 = 8000$, we are obviously looking for a number between 1000 and

8000, but closer to the former.

$1728 \div 6^2$ can be worked out to be 48 from $1728 \div 6 = 288$ by dividing by 6 again, from $1728 \div 12 = 144$ by dividing by 3 or from$1728 \div 12^2 = 12$ by multiplying by 4. Alternatively, we can use the fact that 1728 = 23 ×63 to see that the answer is $2^3 \times 6 = 48$.

ODD OR EVEN?

When n is a positive whole number which of these expressions are:

- always odd?
- sometimes odd and sometimes even?
- always even?

2n + 4

3n + 2

2n + 3

3n + 3

More Creative Use of Odd Moments
Mathematical Association

For many students it will be instructive to create a table of values, possibly using a spreadsheet, to display some values of the expressions. This shows readily which of the three categories each expression comes into and provides a focus for discussing how this is determined by the form of the expression.

n	$2n+4$	$3n+2$	$2n+3$	$3n+3$
0	4	2	3	3
1	6	5	5	6
2	8	8	7	9
3	10	11	9	12
4	12	14	11	15
5	14	17	13	18
6	16	20	15	21
7	18	23	17	24
8	20	26	19	27
9	22	29	21	30
10	24	32	23	33

$2n + 4$ can be written as $2(n + 2)$ and must therefore always be even.

$3n + 2$ gives a sequence that alternates between odd and even. When n is even $3n + 2$ is even, because $3n$ is even, and when n is odd the expression is odd.

$2n + 3$ is always odd because the expression consists of the sum of a number that is always even, $2n$, and an odd number.

$3n + 3$behaves in a similar way as $3n + 2$, but we should also observe that it gives multiples of 3.

SAVING TIME

- How long does it take to go 5 miles at 50mph?
- How much time is saved by doing the same distance at 60 mph?
- How long does it take to go 5 miles at 30mph?
- How much time is saved by doing the same distance at 40 mph?
- How long does it take to go 5 miles at *x* mph?
- How much time is saved by doing the same distance at *x* +10 mph?

The time to go 5 miles at 50mph is found as $\frac{5}{50}$ or $\frac{1}{10}$ of an hour, which is 6 minutes. At 60mph, the time for 5 miles is 5 minutes either by using the useful fact that 60 mph is a mile per minute or from $\frac{5}{60}$ or $\frac{1}{12}$ of an hour. The time saved by increasing the speed by 10mph is 1 minute.

At 30 mph and 40mph, the respective times taken are 10 minutes and 7½ minutes, a saving in this case of 2½ minutes for an increase in speed of 10 mph. Clearly the time saving is not the same for a given increase in speed and is greater when the speeds involved are lower.

To go 5 miles at x mph and x +10mph the respective times are $\frac{5}{x}$ mph and $\frac{5}{x+10}$ mph. The time saved is then given by:

$$\frac{5}{x} - \frac{5}{x+10} = \frac{5(x+10) - 5x}{x(x+10)} = \frac{50}{x(x+10)}$$

It is worth checking that this gives the correct saving for the two examples considered initially. Sketching a graph of $y = \frac{50}{x(x+10)}$ will show that it is a decreasing function for positive values of x. This confirms the observation that the time saved for a given increase in speed decreases as the initial speed increases.

The question arose when I wondered how much time I saved by increasing my speed from 60 to 70mph on a 1 mile stretch of dual carriageway. The answer is about 8½ seconds!

A SURPRISING PATTERN

1				
1	**1**			
1	**2**			
1	**3**	**1**		
1	**4**	**3**		
1	**5**	**6**	**1**	
1	**6**	**10**	**4**	
1	**7**	**15**	**10**	**1**

Can you generate the next three rows?
What is the sum of each row?

What do you notice?

More Creative Use of Odd Moments
Mathematical Association

The next three rows are:

1	8	21	**20**	5	
1	9	28	35	**15**	1
1	10	36	56	**35**	6

Successive lines can be generated by adding diagonal pairs of numbers to find the number below as shown in bold above for one particular case.

The sums of the rows give the Fibonacci sequence:

1 1 2 3 5 8 13 21 34 55 89 144.

In fact the array of numbers is Pascal's triangle with the numbers in the rows displaced. Alternate lines which give the Fibonacci sequence are shown in bold below. The appearance of the sequence is a direct consequence of the addition property that generates Pascal's triangle. We can see that any of the lines of numbers that give the sequence are generated by adding numbers in the two parallel lines above.

1
1 1
1 **2** 1
1 3 **3** 1
1 **4** 6 **4** 1
1 5 **10** 10 **5** 1
1 **6** 15 **20** 15 **6** 1
1 7 **21** 35 **35** 21 7 1

FAREY SEQUENCES

$$0 \quad \frac{1}{2} \quad 1$$

$$0 \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad 1$$

$$0 \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad 1$$

These three sequences are Farey Sequences of orders 2, 3 and 4. Find the Farey Sequence of order 5 by inserting fifths in the last sequence.

Work out some more Farey Sequences.

What patterns do you notice in the sequences?

Here are the Farey sequences of orders 5, 6 and 7:

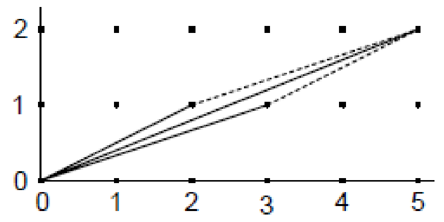
$$0 \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \quad 1,$$

$$0 \quad \frac{1}{6} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \quad \frac{5}{6} \quad 1,$$

$$0 \quad \frac{1}{7} \quad \frac{1}{6} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{2}{7} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{3}{7} \quad \frac{1}{2} \quad \frac{4}{7} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{5}{7} \quad \frac{3}{4} \quad \frac{4}{5} \quad \frac{5}{6} \quad \frac{6}{7} \quad 1.$$

Inserting the fifths and then the sixths is fairly straightforward using a combination of knowing the relative size of unit fractions, the symmetry of the arrangement of the fractions and their decimal forms. The sevenths need a bit more calculation – and there are more of them to be inserted!

Farey sequences are rich in surprising patterns. For instance, for any set of three consecutive fractions, the middle fraction is equivalent to the fraction whose numerator and denominator are the respective sums of the numerators and denominators of the two outer fractions. Indeed adding numerators and denominators of a pair of fractions always gives a fraction that lies between. This can be explained by representing each fraction as a gradient. The diagram shows that two fifths lies between a half and a third.



FOUR CONSECUTIVE NUMBERS

5 6 7 8

Multiply the two outer numbers and the two inner numbers and subtract the smaller from the larger.

Try it with some more sets of four consecutive numbers. Explain what happens.

What happens if you do the same thing with sets of consecutive odd numbers? What about consecutive even numbers?

Try the same idea with sets of four consecutive multiples of 3 or 4.

Prove a general result.

A few examples will rapidly convince students that the answer for any set of consecutive numbers is always two.

$$5\ 6\ 7\ 8 \to 6\times 7 - 5\times 8 = 2$$

$$10\ 11\ 12\ 13 \to 11\times 12 - 10\times 13 = 2$$

$$98\ 99\ 100\ 101 \to 99\times 100 - 98\times 101 = 2$$

This is easily explained algebraically with the four consecutive numbers denoted by n, $n+1$, $n+2$ and $n+3$.

$(n+1)(n+2) - n(n+3) = (n^2+3n+2) - (n^2+3n) = 2$.

With consecutive odd numbers it is convenient to represent the numbers as $2n-3$, $2n-1$, $2n+1$ and $2n+3$. This time we see that the result is always eight.

$$(2n-1)(2n+1) - (2n-3)(2n+3) = (4n^2-1) - (4n^2-9) = 8 .$$

Four consecutive even numbers can be represented as $2n$, $2(n+1)$, $2(n+2)$ and $2(n+3)$. Note the direct comparison to the argument with four consecutive numbers which shows how the result becomes eight.

$$4(n+1)(n+2) - 4n(n+3) = 4(n^2+3n+2) - 4(n^2+3n) = 8 .$$

The argument above can be easily adapted for any set of multiples, but, more generally, if the difference between each pair of consecutive terms is d, we can show that the difference between the two products is $2d^2$.

$$(n+d)(n+2d) - n(n+3d) = (n^2+3dn+2d^2) - (n^2+3dn) = 2d^2$$

TWO-WAY SEQUENCES

Add 2 →

Add 3 ↓

5					

What number goes in the bottom right square?

Change the number in the top left hand square.

Change the numbers you are adding.

Change the operation from addition.

Add 2 →

Add 3 ↓

5	7	9	11	13	15
					18
					21
					24
					27

Two sequences lead to the required number. The table shows one possible route – across and down. The solution can, of course, be reasoned out without recourse to finding all the intermediate numbers and that requires a double application of the procedure for finding a particular term in a sequence.

Starting at 5 and adding 2: 6th term $5 + 5 \times 2 = 15$.

Starting at 15 and adding 3: 5th term $15 + 4 \times 3 = 27$.

In general, for an m by n grid with add a across and add b down and a starting number x, the number in the bottom right corner is given by $x + a(m-1) + b(n-1)$. With a square n by n grid this takes the simpler form: $x + (a + b)(n-1)$. This is illustrated by the diagonal sequence shown below.

Add 2 →

Add 3 ↓

5				
	10			
		15		
			20	
				25