Algorithms and Probability (FS2025)
Week 11

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May 7, 2025

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1 Mini Quiz, Exercises, Corrections

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2 Content

2.1 Finding Arbitrarily Long Paths

Definition: Long Paths: Given a graph G and a number B, find out if there exists a path of length at least B in G.

Theorem: If we can solve the long paths problem for graphs on n vertices in $t(n)$ time, then we can solve the hamiltonian cycle problem in $t(2n-2)+O(n^2)$ time

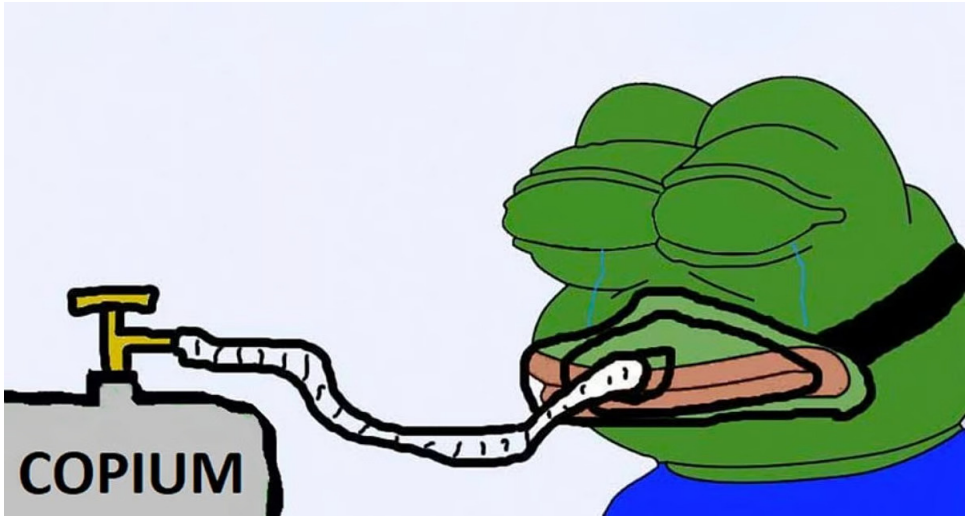


Figure 1: ambitious CS students realizing that we literally cannot have anything good in computer science because of the P = NP Problem.

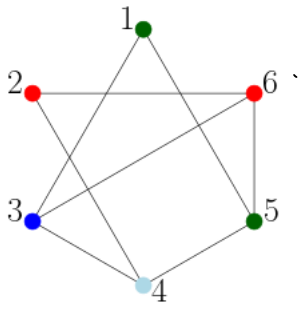
2.2 Copium: Finding Kinda, Sorta Long Paths

After learning that we likely won't come up with an efficient solution to Long Paths, we go through the 5 stages of grief. Instead, we look at the following:

Definition: Kinda, Sorta Long Paths: Given a graph G find out if there exists a path of length at least $\log n$ in G.

Our idea is the following: We will develop a probabilistic algorithm which does multiple iterations of the following:

Quiz / Example:



Which of the following are colorful paths:

- 2,4,5
- 4,2,6
- 1,5,4
- 3,6,5
- 1,2,3
- 1,3,6
- 4,2,5
- 5,4,2
- 2,3,4
- 3,5,6

First, let's find an algorithm for finding colorful paths in a colored graph. Let us fix a number k. Given a graph $G = (V, E)$ and a coloring $\gamma : V \to [k]$ we want to find colorful paths and determine in particular if there exists a path of length $k - 1$ (edges). To this end, we will use the following DP algorithm (yay, your favorite.)

Definition: DP-Definition: Let $P_i(v)$ be our DP entry defined as:

$$P_i(v) := \left\{ S \in \binom{[k]}{i+1} \mid \exists \text{ a colorful path which ends in } v \text{ and traverses exactly the colors in } S \right\}$$

And we define our base cases and recursion as:

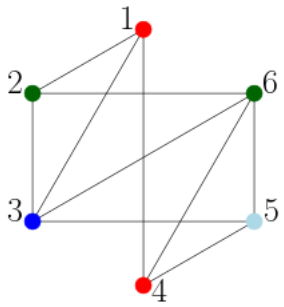
Figure 2: Pseudocode for the BUNT algorithm

Definition:

- Base Case:
- Recursion:
- Solution Extraction:

.

Quiz / Example:



What is $P_2(1)$ in this graph with these colors?

- $P_2(1) = \{\{b, g, r\}, \{g, g, r\}, \{b, c, r\}, \{c, r, r\}\}$
- $P_2(1) = \{(r, g, b), (r, g, g), (r, b, g), (r, b, c), (r, r, c), (r, r, g)\}$
- $P_2(1) = \{\{b, g, r\}, \{b, c, r\}\}$

- $P_2(1) = \{(r, g, b), (r, b, g), (r, b, c)\}$

Runtime Analysis "BUNT":

Theorem: The runtime of "BUNT" is $O\left(\binom{k}{i} \cdot i \cdot m\right)$ and the total runtime of the algorithm is

$$O\left(\sum_{i=1}^{k-1} \binom{k}{i} \cdot i \cdot m\right) \leq O\left(2^k km\right)$$

where we used the equality $\sum_{i=0}^{k} \binom{k}{i} = 2^n$

We are still not done, we proposed this algorithm with the idea in mind that we color the graph somehow and run this previous algorithm. Thus, our idea is the following: we assign colors randomly and independently to every vertex in each round / iteration with $k = \log n + 1$ and then apply the algorithm with the same k. Each iteration will take:

- $O(n)$ time to color the vertices randomly
- $O\left(2^{\log n}(\log n)m\right)$ (dominant factor)

Now, how often do we want to repeat this in total?

$$p_{error} = 1 - \frac{k!}{k^k} \leq 1 - e^{-k}$$

Applying Theorem 2.74:

Monte Carlo?
Las Vegas?

$$O(N \cdot 2^k km) = O(N 2^{\log n}(\log n)m) = O(\lambda(2e)^{\log n}(\log n)m)$$

Quiz / Example:

This full algorithm we have shown either returns "yes" or "no":

- Output "yes" is always correct
- Output "no" is always correct

The BUNT subprocedure we have shown also either returns "yes" or "no":

- Output "yes" is always correct
- Output "no" is always correct

Assume we now want to apply this algorithm differently. We want to find paths of length $k = 2 \cdot \log n$ instead and we will repeat the random coloring process $N = n \cdot e^k$ times What is the runtime and the error probability for this algorithm then? log is binary log

2.3 Flow

Definition: A Network is a tuple $N = (V, A, c, s, t)$, where

- (V, A) is a directed graph (where the edges are the pipes and the vertices are the intersections)
- $s \in V$ is the source (the point where water magically spawns out of)
- $t \in V \setminus \{s\}$ is the target (the point where water disappears into the abyss)
- $c : A \to \mathbb{R}_0^+$ is the capacity function (which denotes how much water can flow through each pipe)

Definition: Let $N = (V, A, c, s, t)$ be a network. A flow in N is defined as a function $f : A \to \mathbb{R}$, such that

- For all $e \in A$, $0 \leq f(e) \leq c(e)$
- For all $v \in V \setminus \{s, t\}$ we have

$$\sum_{u \in V \text{ s.t. } (u,v) \in A} f(u, v) = \sum_{u \in V \text{ s.t. } (v,u) \in A} f(v, u)$$

  which we call conservation of flow (note that again, this does not apply for the source and the target)

Definition: The value of a flow f is defined as

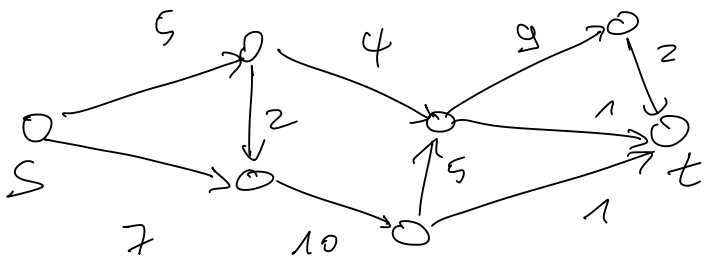
$$\text{val}(f) := netoutflow(s) := \sum_{v \in V \text{ s.t. } (s,v) \in A} f(s, u) - \sum_{u \in V \text{ s.t. } (u,s) \in A} f(u, s)$$

Definition: f is called an integer flow, iff $f(e) \in \mathbb{Z}\ \forall e \in A$

Lemma:

$$\text{netinflow(t)} := \sum_{u \in V \text{ s.t. } (u,t) \in A} f(u, t) - \sum_{v \in V \text{ s.t. } (t,v) \in A} f(u, s) = \text{netoutflow}(s)$$

Examples:



Proof. We can alternatively prove that

$$\begin{aligned}
0 &= \text{netoutflow}(s) - \text{netinflow}(t) \\
&= \left(\sum_{v \in V \text{ s.t. } (s,v) \in A} f(s,v) - \sum_{u \in V \text{ s.t. } (u,s) \in A} f(u,s) \right) \\
&\quad - \left(\sum_{u \in V \text{ s.t. } (u,t) \in A} f(u,t) - \sum_{v \in V \text{ s.t. } (t,v) \in A} f(u,s) \right) \\
&= \sum_{v \in V} \left(\sum_{u \in V \text{ s.t. } (v,u) \in A} f(v,u) - \sum_{u \in V \text{ s.t. } (u,v) \in A} f(u,v) \right) && (\star) \\
&= \sum_{(v,u) \in A} f(v,u) - \sum_{(u,v) \in A} f(u,v) && (\star\star) \\
&= 0
\end{aligned}$$

□

Definition: An s-t-Cut for a network (V, A, c, s, t) is a partition (S, T) of V ($S \uplus T = V$) such that $s \in S$ and $t \in T$. The capacity of this cut is defined as

$$\text{cap}(S,T) := \sum_{(u,w) \in (S \times T) \cap A} c(u,w)$$

↳ only edges going from S to T!

We have that

Lemma: Let f be a flow and (S, T) an s-t-Cut in a network. Then we have that

$$\text{val}(f) \leq \text{cap}(S,T)$$

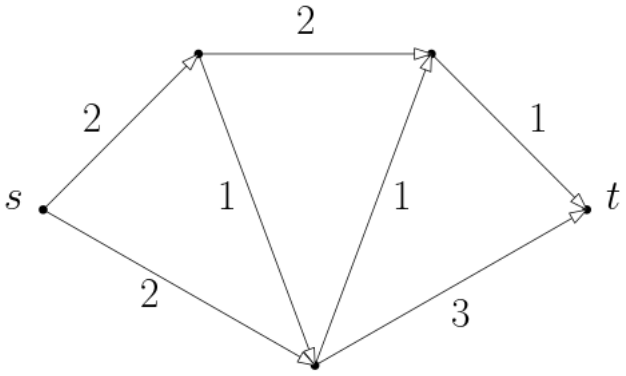
and with these definitions comes the formal version of the

Definition: (Maxflow-Mincut-Theorem) For every network (V, A, c, s, t), we have that

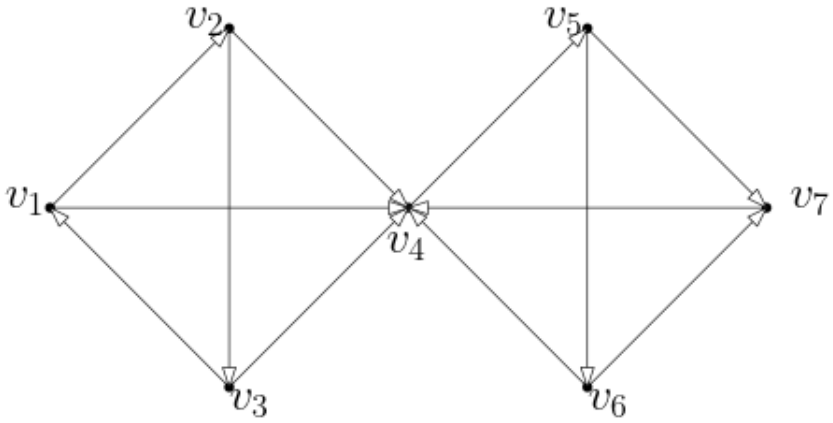
$$\max_{f \text{ a flow in } N} \text{val}(f) = \min_{(S,T)\ s\text{-}t\text{-Cut in } N} \text{cap}(S, T)$$

Quiz Time!

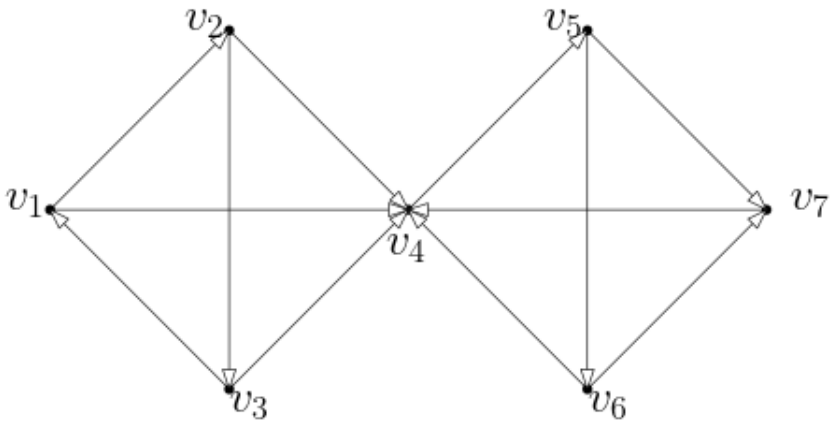
- $N = (V, A, c, s, t)$ be a network and S and T and s-t-Cut. We have that $cap(S, T) \geq 0$
- What is the maximum flow in the following network?



- What is the capacity of the following network with $S = \{v_1, v_2, v_3, v_4\}$, $T = \{v_5, v_6, v_7\}$



- What is the capacity of the following network with $S = \{v_1, v_4, v_5, v_6\}$, $T = \{v_2, v_3, v_7\}$


v_1
v_2
v_3
v_4
v_5
v_6
v_7