Propositional Logic

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Logic is all about how to conceptualize and concisely write concepts and statements, and make inference/draw conclusions from them. Logic comes in different flavors. The simplest one being Propositional Logic (PL). The building blocks of PL are propositions and connectives.

Definition 1. ***Proposition****: A proposition is a declarative statement that's either true or false. For example, "it is hot today", "2+3=4", "if it is hot then we go swim" are propositions.*

In general, statements that are questions, commands, or opinions are not propositions. For example "Hello!", "how are you", "give me the key", are not propositions.

The **building blocks** of PL are propositions and connectives. The simplest form of a proposition is called an **atomic proposition**. We use lower-case letters to denote atomic propositions, such as p, q, r, t, s, etc.

In natural language, we make statements and combine statements with words, such as "and," "or," "if-then," "but," etc. In natural language, we make statements and combine them with words, such as "and"', "or," "if-then," "but," etc. Likewise, in PL, we create compound propositions by combining atomic propositions using logical connectives.

1 Truth Tables

Before we review the major connectives, we define a useful table that allows us to evaluate the truth of a compound proposition based on the values of its atomic propositions.

- **Negation:** Let p be a proposition. We call "not p" the negation of p, denoted by $\neg p$.

p	$\neg p$
T	F
F	T

- **Conjunction:** Let p and q be two propositions. We call conjunction the compound proposition p and q, denoted by $p \wedge q$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- **Disjunction:** Let p and q be two propositions. We call disjunction the compound proposition p or q, denoted by $p \vee q$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- **Implication (if-then):** We can implication the compound proposition "if p then q", denoted $p \rightarrow q$ or $p \Rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- **If and only if (IFF):** Let p and q be two propositions. We call an equivalence the compound proposition "p if and only if q", denoted as $p \leftrightarrow q$ or $p \Leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Just like arithmetic operators, there is a precedence order when evaluating logical connectives as follows (from highest to lowest): $()$ processed from inside to outside, $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, left to right.

Definition 2. ***Tautology***: *A tautology is a proposition that is always true for any assignment of truth values to its atomic propositions.*

Definition 3. ***Fallacy/Contradiction***: *A fallacy (or a contradiction) is a proposition that is always false for any assignment of truth values to its atomic propositions.*

Definition 4. ***Logical Equivalence***: *Two propositions p and q are logically equivalent if and only if they have the same truth tables.*

2 Laws of Propositional Logic

1. **Identity law**:

   $p \wedge True \equiv p$

   $p \vee False \equiv p$

2. **Domination law**:

   $p \vee True \equiv True$

   $p \wedge False \equiv False$

3. **Idempotence law**:

   $p \vee p \equiv p$

   $p \wedge p \equiv p$

4. **Negation law**:

   $p \wedge (\neg p) \equiv False$

   $p \vee (\neg p) \equiv True$

5. **Double negation law**:

   $\neg\neg p \equiv p$

6. **Commutativity law**:

   $p \wedge q \equiv q \wedge p$

   $p \vee q \equiv q \vee p$

7. **Associativity law**:

   $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

   $(p \vee q) \vee r \equiv p \vee (q \vee r)$

8. **Distributivity law**:

   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

   $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

   $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$

9. **Absorption law**:

   $p \vee (p \wedge q) \equiv p$

   $p \wedge (p \vee q) \equiv p$

10. **DeMorgan's law**:

    $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$

    $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$

11. **Implication to disjunction law**:

    $p \rightarrow q \equiv \neg p \vee q$

12. **IFF to implication law**:

    $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

3 Logical Inference

Definition 5. ***Inference/Argument***: *An inference in Propositional Logic is a sequence of propositions denoted as:*

$$\frac{p_1 \; p_2 \ldots p_n}{q}$$

where $p_1, \; p_2, \ldots, p_n$ are called ***premises*** *and q is called* ***conclusion***.

Definition 6. *The inference is called* ***valid*** *if premises $p_1, \ldots, p_n$ implies q, in other words*

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q \text{ is a tautology}$$

Inference and implication:

An inference rule is an implication that is always true (tautology).

Example 1:

$$\frac{\neg\neg p}{p} \text{ is a valid inference.}$$

We can prove this using a truth table.

$\neg\neg p$	p	$\neg\neg p \to p$
T	T	T
F	F	T

3.1 Rules of Inference

Rules of inference

Addition $\dfrac{p}{p \lor q}$	**Conjunction** $\dfrac{p \quad q}{p \land q}$	**Simplification** $\dfrac{p \land q}{p}$	**Disjunctive syllogism** $\dfrac{p \lor q \quad \neg p}{q}$
Hypothetical syllogism $\dfrac{p \to q \quad q \to r}{p \to r}$	**Resolution** $\dfrac{p \lor q \quad \neg p \lor r}{q \lor r}$	**Modus ponens** $\dfrac{p \quad p \to q}{q}$	**Modus tollens** $\dfrac{\neg q \quad p \to q}{\neg p}$

Example 2: Use of modus ponens:

$$\frac{\text{If you forget your jacket then you will be cold} \quad \text{You forgot your jacket today}}{\text{You will be cold}}$$

$$\frac{p \to q \quad p}{q}$$