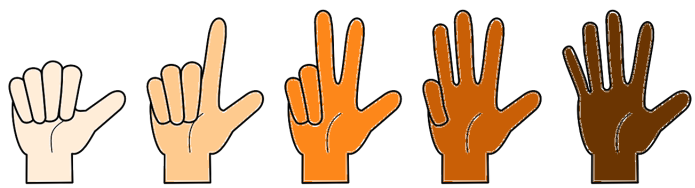
Counting

Aiphabet - aiphabet.org



Counting or combinatorics is a branch of mathematics that deals with counting. Counting skills are essential in many applications, such as computer science and artificial intelligence. We will cover formulas for counting lists using the multiplication theorem, counting sets through combinations, and the Pascal triangle and its properties. We will conclude with the Principle of Inclusion-Exclusion (PIE), which provides ways to calculate the size of the union of sets.

1 Counting Lists

Motivating examples:

1. If you have 5 shirts, 3 pants and 2 pairs of shoes, how many outfits do you have in total? $5 \times 3 \times 2 = 30$

2. How many 2-digit numbers between 10 and 99 are odd? $9 \times 5 = 45$

3. How many phone numbers with 10 digits are there:

   - With repetitions: $10 \times 10 \ldots 10 = 10^{10}$.
   - Without repetitions: $10 \times 9 \times \ldots \times 1 = 10!$.

4. How many 10-digit phone numbers starting with 646 (with repetitions allowed)? 10^7.

Think boxes!

A rule of thumb when counting lists is to use k boxes to model the counting problem. If you make lists of size k, use k boxes and fill each box with the number of possibilities for each box. Then, the multiplication theorem is used to get the total number of possibilities.

Theorem 1 (Multiplication Principle)**.** *The number of lists of length k that can be chosen from a pool of n possible elements is*

$$\begin{cases} n^k & \textit{if repetition is allowed} \\ (n)_k & \textit{if repetition is not allowed} \end{cases}$$

where $(n)_k$ is called **fallen factorial**. $(n)_k = n \times (n-1) \times \cdots \times (n-k+1)$.
We can also write: $(n)_k = \frac{n!}{(n-k)!}$

Permutations

The number of arrangements of k distinct elements chosen from a set of n distinct elements is also called **permutations**.

1.1 Anagrams

Definition 1. ***Anagrams:*** *They are permutations of letters in a word with no repetitions (Nonsensical words are allowed).*

Examples:

1. How many anagrams are in the word math?

math	maht	mtah	mtha	mhta	mhat
amth	amht	athm	atmh	ahmt	ahtm
tamh	tahm	tham	thma	tmah	tmha
hamt	hatm	hmta	hmat	htam	htma

   $\#anagrams(math) = (4)_4 = 4! = 24$

2. How many anagrams in the word: moma?
   $\#anagrams(moma) = \frac{4!}{2!} = 12$. We divide by $2!$ to remove the duplicates (as m appears twice in the word).

3. How many anagrams in the word: intelligence?
   Each of the letters i,n, l occur twice. Letter e occurs 3 times and hence:
   $\#anagrams(intelligence) = \frac{12!}{3!2!2!}$

2 Combinations

Definition 2. ***Combination:*** *We call combinations the k-element subsets of an n-element set for $n, k \in \mathbb{N} \cup 0$ for $0 \leq k \leq n$.*

Definition 3. ***Binomial Coefficients:*** *Let $n, k \in \mathbb{N}$, binomial coefficients denotes by $\binom{n}{k}$ (read n choose k) is the number of k-element subsets of an n-element set.*

It is the combinations of n things taken k at a time.
Other notations: C_k^n, $C(n,k)$. We adopt the $\binom{n}{k}$ notation.

Example:
Let $S = \{a, b, c, d\}$. Suppose we would like to count the number of ways to pick two elements out of the four elements. Here $k = 2$ and $n = 4$.

1. We care about how we arrange the two elements:
   Then we are counting the following possible permutations.

   ```
   ab ac ad
   ba bc bd
   ca cb cd
   da db dc
   ```

   The number of possibilities is: $4 \times 3 = 12$

2. We don't care about how we arrange the two elements:
   Then we are counting the following possible combinations.

   ```
   {a,b} {a,c} {a,d}
   {b,c} {b,d}
   {c,d}
   ```

   The number of possibilities is: 6 sets. We obviously have more permutations than combinations. For each combination, there are $k!$ ways to permute the k elements. For instance for the combination {a,b}, there are two ways to move the elements around to get the lists ab and ba.

Formula for Binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Examples:

1. $\binom{5}{0} = 1$: number of 0-element subsets of a 5-element set.
2. $\binom{5}{1} = 5$: number of 1-element subsets of a 5-element set.
3. $\binom{5}{2} = 10$: number of 2-element subsets of a 5-element set.
4. Note that $\binom{5}{6} = 0$ as we can't make 6-element sets with a 5-element set.

Example:

A mommy rabbit has n baby rabbits and k $(0 \leq k \leq n)$ carrots. In how many ways can mommy rabbit distribute the k carrots?

Either:

- $\binom{n}{k}$: select k lucky baby rabbits, or
- $\binom{n}{n-k}$: select $n-k$ unlucky baby rabbits.

These are two similar counting problems!

Proposition 1.1. *Let $n, k \in \mathbb{N}$, $0 \leq k \leq n$ then:*

$$\binom{n}{k} = \binom{n}{n-k}$$

Examples: Verify that:
$\binom{5}{0} = \binom{5}{5} = 1$
$\binom{5}{1} = \binom{5}{4} = 5$
$\binom{5}{2} = \binom{5}{3} = 10$

2.1 Pascal's Triangle

The coefficients of $(x+y)^n$ form the n^{th} row of Pascal's triangle.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
\vdots
\end{array}$$

Pascal's triangle is generated according to the following four rules:

- Rule 1: The zero row contains only one element, number 1.
- Rule 2: Each successive row contains one more element than the preceding row.
- Rule 3: The first and last element in a row is 1.
- Rule 4: Any intermediate number in a row is formed by adding the two numbers just to its left and right in the previous row.

The entry in row n column k is $\binom{n}{k}$. Rows start at zero and represent n. The columns represented k going from 0 to n.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & \binom{0}{0} & & & & & & & \\
 & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & \\
 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & \\
 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & \\
 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & & \\
 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & \\
 & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} & \\
\binom{7}{0} & & \binom{7}{1} & & \binom{7}{2} & & \binom{7}{3} & & \binom{7}{4} & & \binom{7}{5} & & \binom{7}{6} & & \binom{7}{7} \\
\iddots & & & & & & & \vdots & & & & & & & \ddots
\end{array}
$$

Rule 4 holds thanks to the following Pascal identity:

Theorem 2 (Pascal Identity)**.** *Let* $n, k \in \mathbb{Z}, 0 < k < n$.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Intuition behind the Pascal identity

$\binom{n}{k}$ counts the number of subsets of size k we can make out of a set S of n elements.

Assume S contain a special element, call it the captain. Then the number of subsets of size k we can make out of a set of n elements either contain the captain or do not contain the captain:

- If the captain is in the subsets, then we need to choose $k-1$ other elements among $n-1$ elements (we remove the captain from the possible choices), to make subsets of size k (that must include the captain). The count of these subsets is $\binom{n-1}{k-1}$.
- If the captain is NOT in the subsets, then we need to choose k elements among $n-1$ elements (we remove the captain from the possible choices), to make subsets of size k (that must NOT include the captain). The count of these subsets is $\binom{n-1}{k}$.

So the number of subsets of size k we can make out of n elements is $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

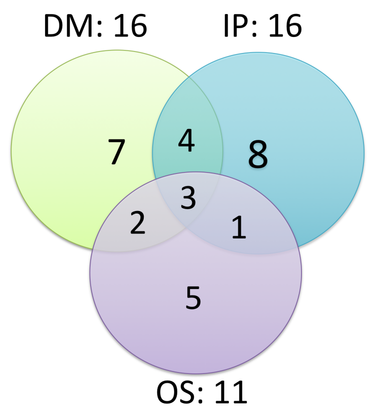
3 Inclusion-Exclusion

In this section, we will learn about formulas to count the size of the union of sets.

Let's take an example.

Example:

Consider the following puzzle: An education counselor is planning classes for 30 students: 16 students say they want to take Discrete Math (DM), 16 want to take Intro to Programming (IP), and 11 want to take Operating Systems (OS). Five say they want to take both DM and OS, and of these, 3 wanted to take IP as well. Five want only OS, and 8 want only IP and 7 only DM.



We would like a formula for the number of students who want to take DM or IP or OS. So we are looking for the cardinality of the union $|DM \cup IP \cup OS|$. The issue is that there are shared elements and we don't have $|DM \cup IP \cup O| = |DM| + |OS| + |P|$. So we will need to correct for adding too many elements by subtracting the pairwise intersections $|DM \cap IP|, |DM \cap OS|$ and $|IP \cap OS|$. When we do that, a new problem arises which is that now we have subtracted too much and removed the intersection three ways $|DM \cap IP \cap OS|$, so we need to add it back.

You can check that:

$$|DM \cup IP \cup OS| = |DM| + |OS| + |P| - |DM \cap IP| - |DM \cap OS| - |IP \cap OS| + |DM \cap IP \cap OS|$$

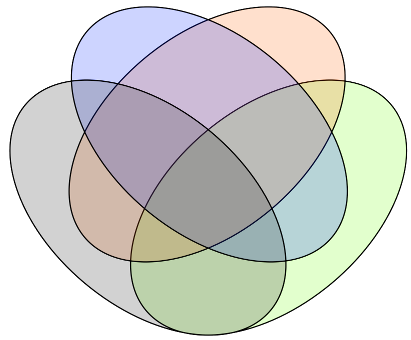
The idea of correcting for the sizes of the intersections works for two, three and four sets as follows.

- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

Last term: add or subtract?

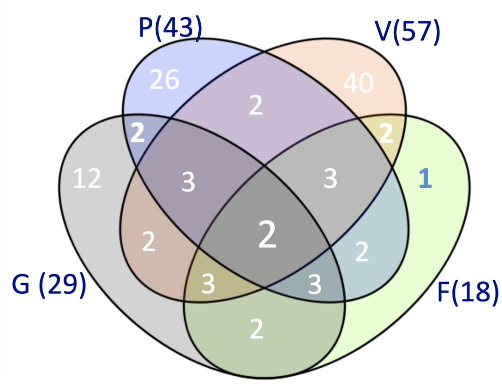
Note that depending on whether the number of sets is even or odd, the last term of the formula is either a subtraction or an addition respectively.

Example: Draw all possible intersections with four sets:



It becomes tedious with more and more sets to use a Venn diagram to figure out the union. Hence, the usefulness of the PIE formula!

You can check that the PIE formula for four sets holds for the following example:



At the music academy, there are 43 students taking piano, 57 students taking violin, 29 students taking guitar, and 18 taking flute. There are 10 students in any two of these courses, 5 students in any three of them and 2 taking all courses. Use the Inclusion-Exclusion principle to find the number of students who are taking at least one course at the music academy.

$|A_p| = 43$, $|A_v| = 57$, $|A_g| = 29$, $|A_f| = 18$. There are $\binom{4}{1} = 4$ such terms.

Intersections pairwise: $|A_i \cap A_j| = 10$. Where $i, j \in \{p, v, g, f\}$ There are $\binom{4}{2} = 6$ such terms

Intersections three ways: $|A_i \cap A_j \cap A_k| = 5$. Where $i, j, k \in \{p, v, g, f\}$ There are $\binom{4}{3} = 4$ such terms.

Intersections four ways:$|A_p \cap A_v \cap A_g \cap A_f| = 2$. There are $\binom{4}{4} = 1$ such terms.

Therefore, $|A_p \cup A_v \cup A_g \cup A_f| = 43 + 57 + 29 + 18 - 10 * 6 + 5 * 4 - 2 = 105$.

It is much easier to use inclusion-exclusion than drawing a Venn diagram.

4 Practice Problems

4.1 Counting Lists

1. How many anagrams can you make out of the word AIPHABET?

2. A baker prepares a dessert for every day of the week starting on Sunday. They can choose to serve cake, pie, ice cream, or cookies. However, the same dessert must not be served two days in a row. Additionally, the baker must serve cake on Wednesday for a birthday. How many different dessert menus are possible this week?

4.2 Combinations

1. What is the coefficient of the x^4y^3 term in the expansion of $(x + 2y)^7$?

2. Toby rolled three 10-sided die. Given that he rolled three distinct values, what is the probability he only rolled prime numbers?

4.3 Inclusion-Exclusion

1. A school has 350 students. 260 like Math, 100 like English, and 70 like Science. Additionally, 40 like Math and Science, 40 like English and Science, and 30 like Math and Science. Given that every student likes at least 1 subject, how many students like all 3 subjects?

4.4 Answers

1. (a) $\frac{8!}{2!} = \boxed{20,160}$

   (b) If the cake is chosen on Wednesday, one of the other 3 desserts must be chosen on Tuesday, as well as Thursday. Whatever those deserts are, one of the other desserts must be chosen on Friday and Monday, and this cascades until the last day. As a result, our answer is simply $3^6 = \boxed{729}$.

2. (a) We have three 2s being multiplied in by the y coefficient in the original expression, plus $\binom{7}{3}$ number of times that the term should appear. This gives us $8 * 35 = \boxed{280}$ as the final coefficient.

   (b) The number of ways Toby can roll three unique numbers is $\binom{10}{3} = 120$. There are 4 prime numbers under 10, so the number of ways Toby can roll three unique $primes$ is $\binom{4}{3} = 4$. Thus, the total probability will be $\frac{4}{120} = \boxed{\frac{1}{30}}$.

3. Using PIE, the answer is $\boxed{30}$