Functions

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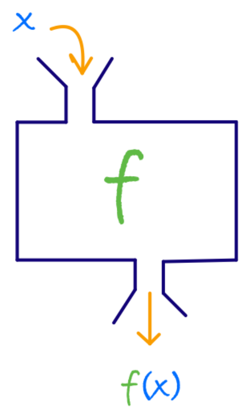


1 Definitions

Definition 1. ***Function***: *Let A and B be two sets. A function f from A to B, denoted $f : A \to B$, is a subset of $A \times B$, i.e. f is a set of pairs (a, b) such that $a \in A$ and $b \in B$ and where* **each element of** *A appears* **exactly once** *as a* **first element** *of an ordered pair.*

- A is called the domain of f
- B is called the co-domain of f
- if $(a, b) \in f$, then we write $b = f(a)$
  - b is the image of a under f
  - a is the pre-image of b under f.
    Note: b could have a set as a pre-image and not only one element. It is acceptable to write the image and pre-image as sets.

Examples:

- $f_1 = \{(1, 2), (2, 3), (3, 1), (4, 7)\}$ is a function ($f_1 : A = \{1, 2, 3, 4\} \to B = \{1, 2, 3, 7\}$).
- $f_2 = \{(1, 2), (1, 3), (4, 7)\}$ is not a function as element 1. appears twice as a first element in f.

- $f_3 : \mathbb{Z} \to \mathbb{Z},\ f_3(x) = x^2$ is a function

Notation:
Let f be a function. f can be represented as:

- a set of pairs, e.g. $f = \{(0,0), (1,2), (2,4), (3,6), \cdots\}$
- a set builder, $f = \{(a,b) | a \in A \wedge b \in B \wedge b = f(a)\}$
- $f : A \to B,\ f(a) =$ some value

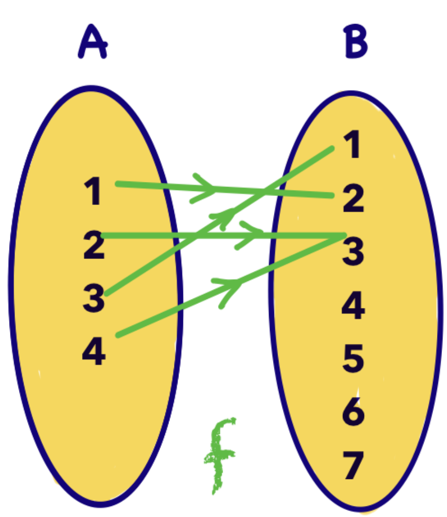
Definition 2. Domain, co-domain, and image*: Given $f : A \to B$ where A is the domain and B is the co-domain,*

- *the set of all possible first elements of pairs of f is called the* **domain** *of f. Let's denote it domain(f).*
- *the set of all possible second elements of pairs of f is called* **image** *of f, denoted image(f).*

Example:
Let f be a function defined on integers, $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = 2x$.
The co-domain of f is $\mathbb{Z}$ and image of f is the set of all even integers which is a subset of the co-domain.

Definition 3. Representation of functions*: We can represent functions with* **bipartite graphs***. A bipartite graph has a vertex set which can be partitioned into two disjoint subsets A and B, such that every edge joins a vertex in subset A to a vertex in subset B. In the context of representing a function f, the sets are the domain and co-domain of f.*

Example:
Say $f : \{1,2,3,4\} \to \{1,2,3,4,5,6,7\}$ where $f = \{(1,2), (2,3), (3,1), (4,3)\}$



Definition 4. Identity function: *The identity function I_A defined on A is the function $I_A : A \to A$, such that:*

$$I_A(x) = x \quad \forall x \in A$$

The identity function sends each element from A to itself. In some sense, you can consider it to behave like a mirror.

2 Properties of functions

Definition 5. Onto/surjective functions: *A function $f : A \to B$ is called* **onto** *provided that every element of B has a pre-image. In other words, image$(f) = B$.*

In the bipartite graph, we check that every vertex in B receives **at least one** edge.

> **How to show a function is onto:**
> To show that a function is onto, choose an arbitrary element $b \in B$, and show that it has a pre-image in A.

Definition 6. One-to-one/injective functions: *A function f is called* **one-to-one** *provided that no element of B is the image of two distinct elements of A using f. In other words, no element in B has more than one pre-image.*

On the bipartite graph, we check that think every element in B receives **at most one** edge from A.

> **How to show a function is one-to-one:**
> To show that s function is one-to-one, assume $f(a_1) = f(a_2)$, and show that $a_1 = a_2$.

Definition 7. Bijective functions or bijections: *A function $f : A \to B$ is called a* **bijection** *provided it is both onto and one-to-one.*

Intuitively, we think of a function being "onto" when every vertex in B receives **at least one** edge. Similarly, a function is one-to-one says when each vertex receives **at most one** edge. Thus, a function is bijective when every vertex in B receives **exactly one** edge.

> **How to show a function is bijective:**
> To show that a function is bijective, show it is both onto and one-to-one.

Exercise: Say $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^3$. Is f a bijection? If yes, prove it. If no, give a counterexample.

Solution:

Onto: $\forall b \in \mathbb{R}, \exists a \in \mathbb{R}$ s.t. $f(a) = b$? We see that $f(a) = b$ means $a = b^{1/3}$ which is still in R (domain). Hence f is indeed onto.

One-to-one: $\forall a_1, a_2 \in \mathbb{R}$, do we have $f(a_1) = f(a_2) \implies a_1 = a_2$? We see, if $f(a_1) = f(a_2)$, it must be $a_1^3 = a_2^3$. That means $(a_1^3)^{1/3} = (a_2^3)^{1/3}$ which is exactly $a_1 = a_2$. So f is one-to-one.

Therefore, f is a bijection.

Note that if we now ask the same question for $f(x) = x^n$, our answer depends on n. We know from the previous exercise, that $f(x) = x^2$ is not one-to-one for $dom\ f = \mathbb{Z} \subseteq \mathbb{R}$. So $f(x) = x^2$ is also not one-to-one for $f : \mathbb{R} \to \mathbb{R}$, hence not a bijection. In general, if n is odd, f is a bijection; if n is even, f is not a bijection.

3 Composition of functions

Definition 8. Composition: *Let the functions $f : A \to B$ and $g : B \to C$. The composition function $g \circ f : A \to C$ is defined as*

$$g \circ f(x) = g(f(x))$$

.

Example: Say $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2 + 1$. $g : \mathbb{Z} \to \mathbb{Z}$ where $g(x) = 2x - 3$. Hence we can compose two functions and have $g \circ f : \mathbb{Z} \to \mathbb{Z}$, where $g \circ f(x) = g(f(x)) = g(x^2 + 1) = 2(x^2 + 1) - 3 = 2x^2 - 1$. As a sanity check, both $g \circ f(2) = 2 \times 2^2 - 1 = 7$ and $g(f(2)) = g(2^2 + 1) = g(5) = 2 \times 5 - 3 = 7$ give the same result.

4 Inverse of functions

Definition 9. Inverse function*: Say we have a function $f : A \to B$. If there exists a function $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$, then g is called the inverse function of f. We write $g = f^{-1}$.*

Example:

Say $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 3x - 2$. To find the inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$, we simply observe $f(x) = 3x - 2 \implies x = \frac{f(x)+2}{3}$. Obviously, if we let $f^{-1}(y) = \frac{y+2}{3}$, we see that it gives the identity function after composition $f^{-1} \circ f(x) = \frac{f(x)+2}{3} = x$. Again, as a sanity check, $f^{-1} \circ f(2) = f^{-1}(f(2)) = f^{-1}(3 \times 2 - 2) = f^{-1}(4) = (4 + 2)/6 = 2$.

Example:

Say we have an encoding function E and decoding function D. Suppose we are sending bits through a channel. To avoid errors from transmission loss, we will triple every digit (why not double only?). For example, 00 will be encoded as 000 000, while 01 will be encoded as 000 111.

We need to be able to recover the correct information using a decode function D. Note that $D = E^{-1}$ so $D \circ E(x) = x$.

5 Pigeonhole principle

Definition 10. Pigeonhole Principle (PHP): *If we have n pigeons living in m holes with $n > m$, then at least one of the holes contains two or more pigeons. More formally, a function $f : A \to B$ cannot be one-to-one if $|A| > |B|$.*

Examples:

1. Suppose we have a group of 13 students. Explain why there must be at least 2 students with same birth month.

   By PHP, we know that since there are twelve months in a year, at least 2 students must share the same birthday month. Once again, in this example, there are more pigeons than holes.

2. In NYC, two people must have same number of hair strands. There are 8 million people in NYC and the number of hair on max 500,000 hair. By pigeonhole principle, two people must have same hair count as there are more pigeons than holes.

3. $A = \{1, 2, \cdots, 8\}$. If 5 integers are selected from A. We must have at least one pair with sum 9.

   1+8=9, 2+7=9, 3+6=9, 4+5=9,

   There are four pairs of numbers that add up to 9, and each of the numbers from 1 to 8 belongs to one of these four pairs. Given that we pick five numbers belonging to four pairs, then our five number must contain two numbers from one pair.

4. Suppose we have a group of n people. If each person has at least one friend, then there will always be two people who have an identical number of friends within the group.

   Think each pigeon here as an individual in this groups while a hole is the number of friends each individual has. One cannot be a friend with himself, so at most a one would have $n - 1$ friends. So there are only $n - 1$ holes in this setting. However, we have n pigeons. So it must be there are two people share the same number of friends.

6 Practice Problems

6.1 Functions

1. Which of the following are functions? Justify your answer.

   (a) $f_1 : \mathbb{Z} \to \mathbb{N}$ defined by $f_1(x) = |x|$ (The absolute value of x)

   (b) $f_2 : \mathbb{R} \to \mathbb{Z}$ defined by $f_2(x) = \lfloor x \rfloor$
   The floor function $\lfloor x \rfloor$ associates each real number x with the greatest integer that is less than or equal to x.

   (c) $f_3 : \mathbb{N} \to \mathbb{N}$ defined by $f_3(x) = x - 4$

   (d) $f_4 : \mathbb{Z} \to \mathbb{N}$ defined by $f_4(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x - 1 & \text{if } x < 0 \end{cases}$

   (e) $f_5 : \mathbb{Z} \to \mathbb{N}$ defined by $f_5(x) = x^2 + 1$

   (f) $f_6 : \mathbb{N} \cup \{0\} \to \mathbb{Q}$ defined by $f_6(x) = \frac{1}{x}$

2. Explain why $(x - 2)^2 + (y - 5)^2 = 1$ is not a function.

   For the following two questions, assume that $f : \mathbb{R} \to \mathbb{R}$.

3. Is $f(x) = 7x - 1$ bijective? Prove your answer. If it is bijective, find $f^{-1}(x)$.

4. Is $f(x) = \lfloor x \rfloor$ bijective? Prove your answer. If it is bijective, find $f^{-1}(x)$.

6.2 Pigeonhole principle

1. An octopus has 8 legs. In its drawer, it has an infinite number of socks, each of which is colored, red, orange, blue, or purple. How many socks must the octopus take out of the drawer to ensure that it can wear a same-color sock on each of its legs?

2. Show that in a group of 20 people in which friendship is mutual (if A is friends with B, then B is friends with A), there exist two people with the same number of friends.

6.3 Answers

1. (a) i. Yes!
      ii. Yes!
      iii. No. $f_3(1) \notin \mathbb{N}$.
      iv. Yes!
      v. Yes!
      vi. No. $f_6(0) \notin \mathbb{Q}$.

   (b) Note that $(x - 2)^2 + (y - 5)^2 = 1$ is a circle of radius 1 centered at $(2, 5)$. Inspecting the graph, we see that $(2, 4) and (2, 6)$ are both on the circle. Since one value of x corresponds to more than one value of y, $(x - 2)^2 + (y - 5)^2 = 1$ cannot be a function.

(c) Choose some (x_1, x_2) s.t. $x_1 \neq x_2$. It follows that $7x_1 - 1 \neq 7x_2 - 1$, so $f(x)$ is injective. Next, choose any $b \in \mathbb{R}$. For $a = \frac{b+1}{7}$, $f(a) = b$. Thus, we can say that $f(x)$ is surjective, which completes our proof. $f^{-1}(x) = \frac{x+1}{7}$.

(d) As the floor function's range is only $\mathbb{Z}$, $f(x)$ cannot be surjective. Thus, $f(x)$ is also not bijective.

2. By the GPHP, if we have 29 socks, the octopus must have at least $\lceil \frac{29}{4} \rceil = 8$ socks of the same color. This is also the minimum number of socks necessary - if the octopus only had 28 socks, they could've drawn 7 socks of each color, which fails the condition.

3. Note that as friendship is mutual, it is impossible for one person to have 0 friends while a different person has 19 friends. Thus, there are at most 19 different numbers of friends that can be had in this group, which means that there must be 2 people with the same number of friends by the PHP.