Graph Theory

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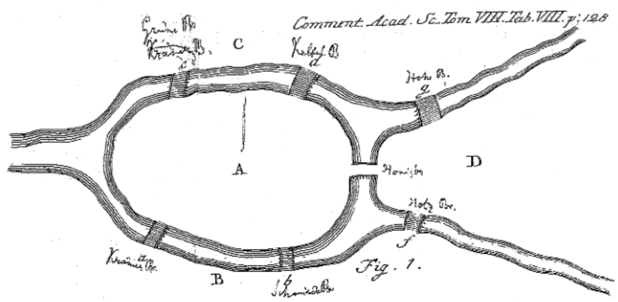
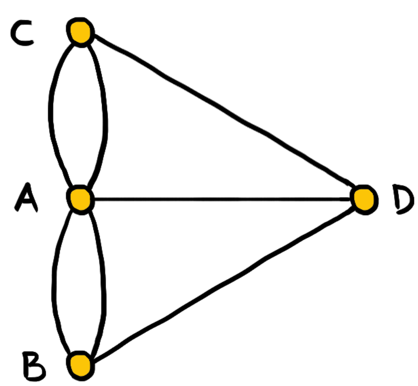


Image from Euler's original paper.

The origins of the field of Graph Theory goes back to Leonhard Euler (read Oiler), who provided a proof of impossibility of the puzzle of the seven bridges in the city of Königsberg. There were different areas A, B, C and D in the city connected by seven bridges. The challenge was to find a tour that crosses all seven bridges exactly once and returns to the starting point.

Euler showed that such a tour is impossible to realize. The problem can be abstracted with the graph below, in which he showed areas A, B, C and D as nodes and the bridges as lines or connections between them. The puzzle can now be formulated as follows: from a starting point, that is a node, start tracing with a pencil the connections and try to return to that starting point without lifting the pencil.



You can try this task by hand but soon you will realize that this is impossible. The problem comes from the fact that there is an odd number of lines that meet at each node. There should be an even number of lines as each time we trace a line to that node, we need to trace out through a different line. Therefore, the number of lines at each node must be even.

Graphs are powerful abstraction tools that useful in all areas of science and computer science, and Artificial Intelligence!

But what is a graph?

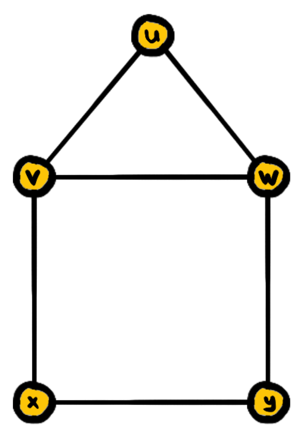
1 Basic Definitions

Graphs are representations of related objects using two sets of objects: vertices and edges. We begin with the formal definitions and terminology for graphs.

Definition 1. ***Graph:*** *A graph $G = (V, E)$ consists of two sets, V, a set of vertices (plural for vertex) and, E, a set of edges. Sometimes, we denote $E(G)$ the set of edges and $V(G)$ the set of vertices.*

Definition 2. ***Edge:*** *An edge joints two vertices called its endpoints. An edge between two vertices u and v is denoted $\{u, v\}$ or uv.*

Example: Below, we show a graph, with vertices {u,v,w,x,y} and edges uv, uw, vw, vx, xy, yw



> **Simple graphs**
> A simple graph is an undirected graph with no loops, no direction, and no parallel edges. We will focus in the following on simple graphs.

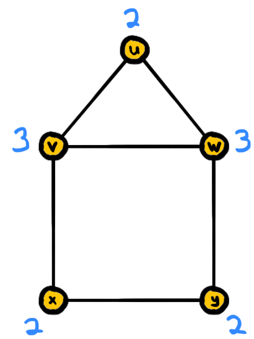
Definition 3. ***Adjacent:*** *Two vertices are adjacent if there is an edge $\{u, v\} \in E$ joining them. We note: $u \sim v$ and also $u - v$. If $u \sim v$ we also say that u and v are neighbors.*

Per this definition, a vertex in a simple graph is never adjacent to itself.

Definition 4. ***Degree:*** *The number of neighbors of a vertex is called its degree, denoted $d(v)$.*

The maximum degree of a graph is denoted $\Delta(G)$ and the minimum degree of a graph is denoted $\delta(G)$.

Example: In the graph above, $d(v) = 2$, $\Delta(G) = 3$ (from vertex x), and $\delta(G) = 1$ (from vertex y). We label below each vertex with its degree.



> **Observe!**
>
> If we sum up the degrees in the graph above, we find exactly twice the number of edges:
>
> $$d(u) + d(v) + d(w) + d(x) + d(y) = 2 + 3 + 3 + 2 + 2 = 2 \times 6$$
>
> That's not a coincidence. It happens for every graph!

Theorem 1 (Handshaking)**.** *The sum of the degrees of each vertex in a graph G is equal to twice the number of edges.*

Proof. Let G be a graph with n vertices and m edges. Each edge in G contributes 2 to the total degree of G. Then, for any arbitrary edge e with endpoints v_i and v_j, e contributes 1 to the degree of v_i and 1 to the degree of v_j. Therefore, e contributes 2 to the total degree of G.

If we consider all the edges, each contributes 2 to the total degree. Hence, we have the sum of the degrees of each vertex in a graph G is equal to twice the number of edges. □

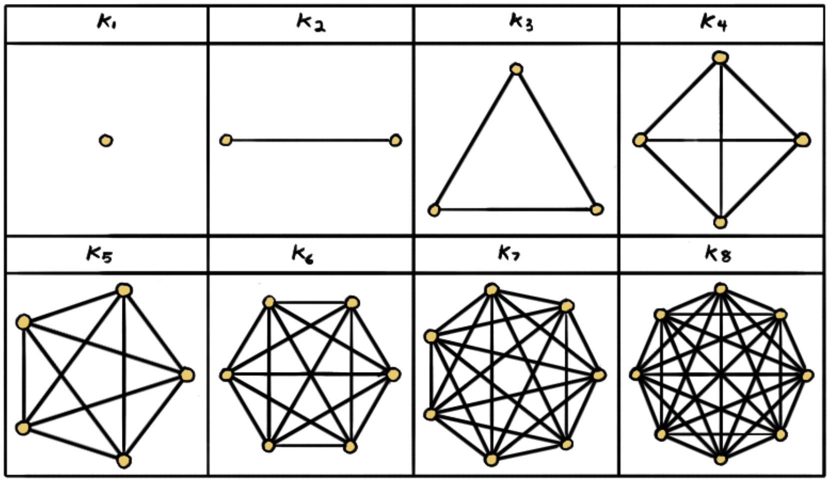
2 Types of Graphs

There are different types of graphs. We examine some of them below.

Definition 5. ***Null graph:*** *A null graph with is a graph with vertices nbut no edges. N_n denotes the null graph on n vertices.*

Definition 6. ***Complete graph:*** *A complete graph is a graph with every pair of vertices joined by an edge. K_n denotes the complete graph on n vertices.*

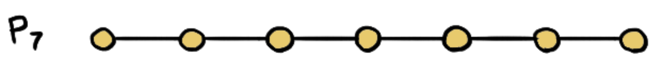
Example: Below are complete graphs for different numbers of vertices and their edge counts.



Source: Wikipedia

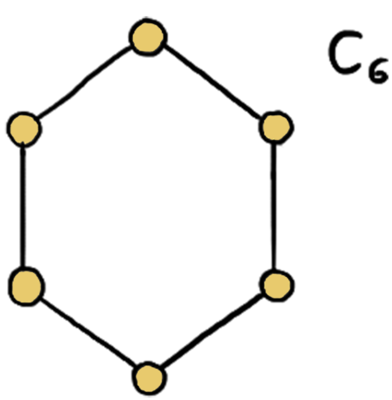
Definition 7. ***Path graph:*** *A path graph has vertices* $\{v_1, v_2, ...v_n\}$ *and edges* $\{e_1, e_2, ...e_{n-1}\}$ *such that edge* e_k *joins vertices* $\{v_k, v_{k+1}\}$*.* P_n *denotes the path graph on* n *vertices.*

Example: Below is a path graph for $n = 7$ vertices.



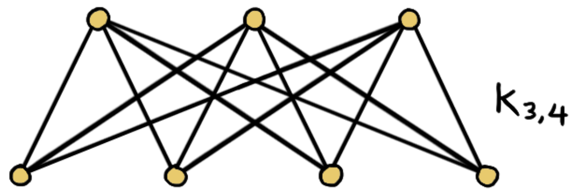
Definition 8. ***Cycle graph:*** *A cycle graph has vertices* $\{v_1, v_2, ...v_n\}$ *and edges* $\{e_1, e_2, ...e_n\}$ *such that edge* e_k *joins vertices* $\{v_k, v_{k+1}\}$ *where* $k + 1$ *is "mod n."* C_n *denotes the cycle graph on* n *vertices.*

Example: The following graph is the C_6 cycle graph.



Definition 9. ***Bipartite Graph (Utility Graph):*** *A bipartite graph has a vertex set which can be partitioned into two disjoint subsets A, B such that every edge joins a vertex in subset A to a vertex in subset B. A complete bipartite graph is a simple graph that is bipartite and every vertex in one subset is joined to a vertex in the other. $K_{m,n}$ denotes the complete bipartite graph with $m + n$ vertices.*

Example: The following graph is a bipartite graph, with the black vertices as set A and the white vertices as set B. Furthermore, this is a complete bipartite, $K, 3.4$.



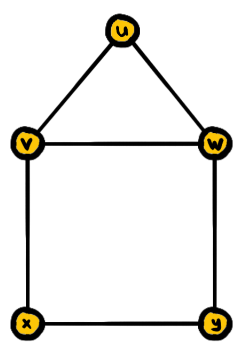
Observe!

Each vertex in C_6 has degree 2 and each vertex in K_5 has degree 4.

3 Trees

Definition 10. ***Walk:*** *A walk in G is a sequence of vertices where each vertex is adjacent to the next vertex.*

Example: In the figure below, (u, v, w, v, x) is a walk of length 4. (v) is a walk of length 0. (u, u, x, v, w) is NOT a walk because u is not adjacent to u.



Definition 11. ***Path:*** *A path in a graph is a walk in which no vertex is repeated. We denote a path of length $n-1$ with n vertices as P_n. A (u,v)-path is a path that starts at vertex u and ends at vertex v.*

Definition 12. ***Connection:*** *Let $G=(V,E)$. We say that u is connected to v provided there is a (u,v)-path in G.*

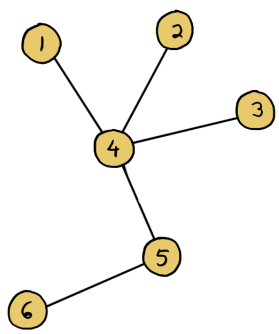
Definition 13. ***Connected Graph:*** *A graph G is called a connected graph provided each pair of vertices is connected by a path.*

Definition 14. ***Tree:*** *T is a tree iff it is connected and has no cycle (acyclic).*

Definition 15. ***Leaf:*** *Let T be a tree. A vertex of degree 1 is called a leaf.*

Definition 16. ***Rooted tree:*** *Let T be a tree. T is said to be rooted provided one of the vertices is chosen to be a root.*

Example: Below is an examples of tree with the leaves 1, 2, 3, and 6.



Theorem 2. *Let $G=(V,E)$. $\forall u,v\in V$ there is exactly one (u,v)-path iff G is a tree. That is, a graph is a tree iff between any two vertices there is a unique path.*

Theorem 3. *Let T be a tree, $T=(V,E)$. Then the number of edges is $|E|=|V|-1$.*

Proof. We prove the above theorem by induction on the number of vertices.

- **Base Case:** For $n=1$, we have that $|V|=1, |E|=0$, therefore $|E|=1-1=0$.
- **Inductive Hypothesis:** Suppose that for a tree T' with k vertices, $|E(T')|=|V(T')|-1$. In other words, $|E(T')|=k-1$.
- **Inductive Step:** We now show for tree T with $k+1$ vertices. It suffices to show that T has k edges. Let v be a vertex in T, where v is a leaf. Thus, v has degree of 1. Consider $T-v$, which from the proposition above is also a tree. $T-v$ has k vertices and, by inductive hypothesis, $k-1$ edges. We deleted one edge of T. Therefore, T has one more edge than T', and $(k-1)+1=k$, which is what we wanted to show.

□

Trees are the least connected graphs

How are trees and complete graphs same and how are they different? Both are connected. However, a complete is maximally connected while a tree is minimally connected. In other words, removing one single edge anywhere in a tree will disconnect it. Every pair of vertices in a complete graph have a path with one single edge, because every pair of vertices are adjacent to each other.

4 Practice Problems

1. There are 25 students in the class. Is it possible that 6 of them have 9 friends each, 8 of them have 8 friends each, and 11 of them have 7 friends each? (friendships are mutual)

2. How many different graphs can we create using the set of vertices $V = \{1, 2, 3, \cdots, n\}$?

3. Among a group of 5 people, is it possible for everyone to shake hands with exactly 2 people? What about 3 people? Prove your answers.

4.1 Solutions

1. Let's formulate this as a graph problem. Let $G = (V, E)$ be a graph with 25 vertices (students), and each edge represents a friendship between two of them. Then, the total degree of the graph would be

$$\sum_{v \in V} deg(v) = 6 * 9, 8 * 8, 11 * 7 = 195.$$

   But by the handshaking theorem, the total degree of the graph must be even, so this arrangement of friendships is not possible.

2. Let's treat this as a counting problem. As each potential edge is between two vertices, there are exactly $\binom{n}{2}$ different possible edges. Since each edge can either exist or not exist, there must be exactly $\boxed{2^{\binom{n}{2}}}$ different possible graphs of this set of vertices.