Calculus

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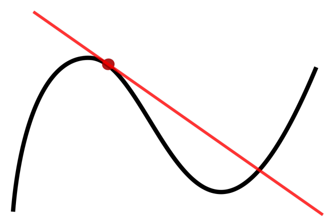


Calculus is a powerful tool that has revolutionized the way we understand the world around us. It allows us to describe and predict the behavior of everything from the tiniest subatomic particles to the largest structures in the universe.

At its core, calculus is all about understanding how things change over time. It helps us to answer questions like "How fast is this object moving?", "What is the rate of growth of this population?", and "What is the slope of this curve?".

One of the most amazing things about calculus is its ability to model complex systems and phenomena. It plays a crucial role in many areas of science, engineering, and technology. It is used to design and optimize machines, to analyze data from experiments, and to understand the behavior of materials and structures.

Overall, calculus is a fascinating and powerful tool that has transformed our understanding of the world around us. It allows us to make accurate predictions, solve complex problems, and design better systems and technologies.

Prerequisites: Algebra, geometry, and trigonometry

1 Rate of Change without Calculus

A slope of a function is defined as the rate of change of the output of a function with respect to the input variable. It tells us how much the output of the function changes for a given change in the input. For example, the slope of a line can tell us the direction and steepness of the line, while the slope of a curve can tell us the curvature and changing rate of the function.

Definition 1 ***Average Rate of Change:*** *The average rate of change between two points is a measure of the average quantity of change over a given interval.*

An average rate of change can be calculated by taking the difference between the final value and the initial value of the quantity and dividing this difference by the length of the interval. In mathematical terms, it is:

$$\underline{\text{Average Rate of Change}} = \frac{\Delta y}{\Delta x} = \frac{y_{final} - y_{initial}}{x_{final} - x_{initial}} = \frac{y_1 - y_0}{x_1 - x_0}$$

We often denote the first/initial point, where the interval or "slice" begins, as the point (x_0, y_0) and the second/final point, where the interval or "slice" ends, as the point (x_1, y_1).

Let's see an example problem with this concept:

Example 1 *Polynomial Functions - Average Velocity*

Suppose you are given the position function $s(t) = 3t^2 - 2t + 1$, which represents the position of a particle at time t. What is the average velocity of the particle between $t = 2$ and $t = 4$?

First, let us note that average velocity is defined as the average rate of change of positions between a certain time interval. In mathematical terms:

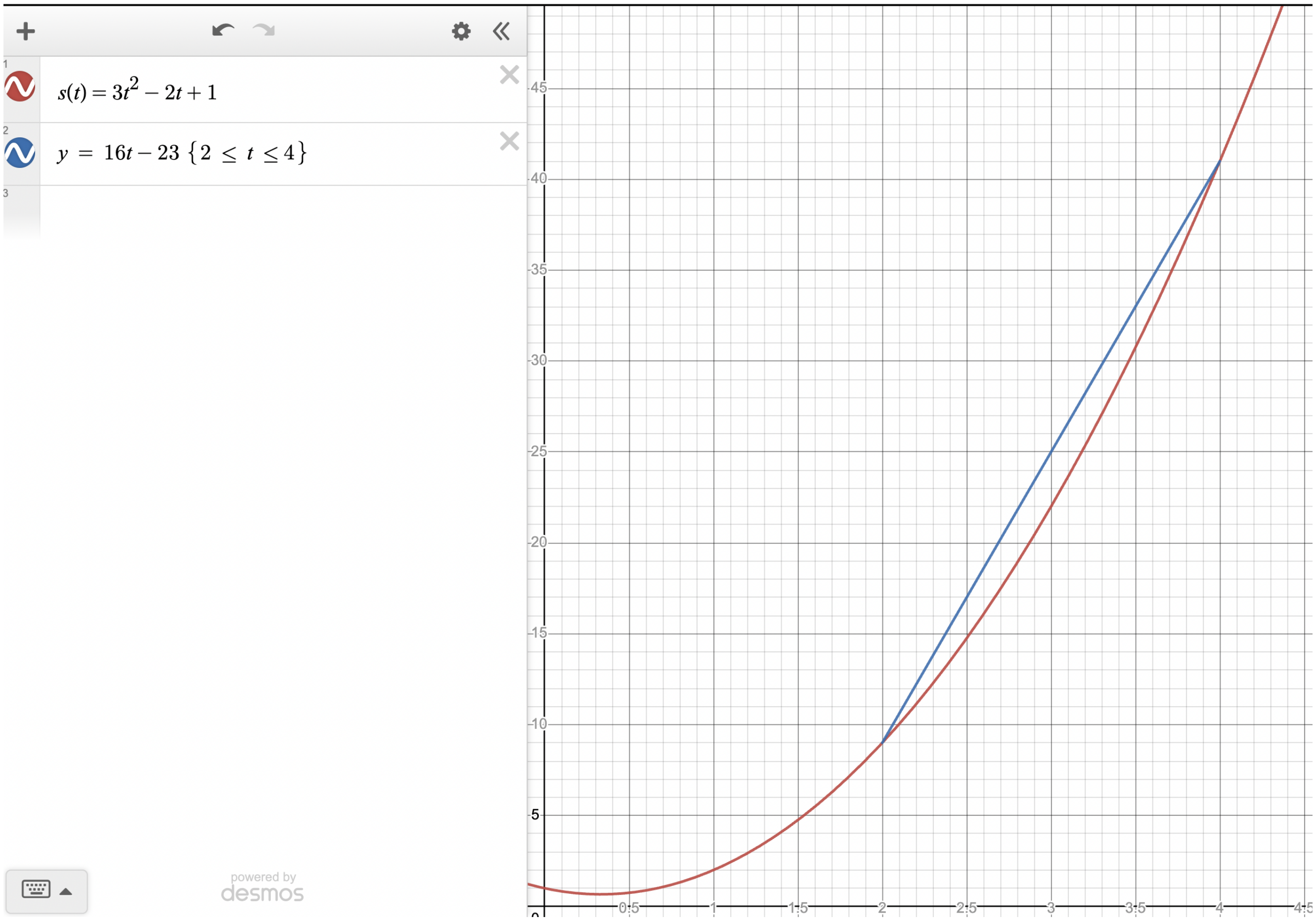
$$v = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

Therefore, to solve this problem, we need to find the difference in position of the particle between $t = 4$ and $t = 2$, and divide that by the difference in time between the two points:

$$\begin{aligned}\underline{\text{Average Rate of Change}} &= \frac{\Delta y}{\Delta x} = \frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{s(4) - s(2)}{4 - 2} \\ &= \frac{(3 \cdot 4^2 - 2 \cdot 4 + 1) - (3 \cdot 2^2 - 2 \cdot 2 + 1)}{4 - 2} = \frac{32}{2} = 16\end{aligned}$$

Therefore, the average velocity of the particle between $t = 2$ and $t = 4$ is 16 units per second, which means that the particle moves at an average rate of 16 units per second over this time interval.

Visually, this is what our calculation looks like (plotted in Desmos):



The red line is our positional function and the blue secant line is the average velocity we calculated. Note that after inducing a y-offset shift, the average rate of change will pass through both the initial and final points we used in the calculation.

2 Limits

Here in Section 2, we will take a bit of a detour to describe what a limit is and why it is useful. Then, in Section 3, we will revisit average rates of change (Section 1) and will combine them with limits in order to find the average rate of change as the initial point nears the final point.

Definition 2 ***Limit:*** *A limit is the value that a function approaches as the input approaches some value.*

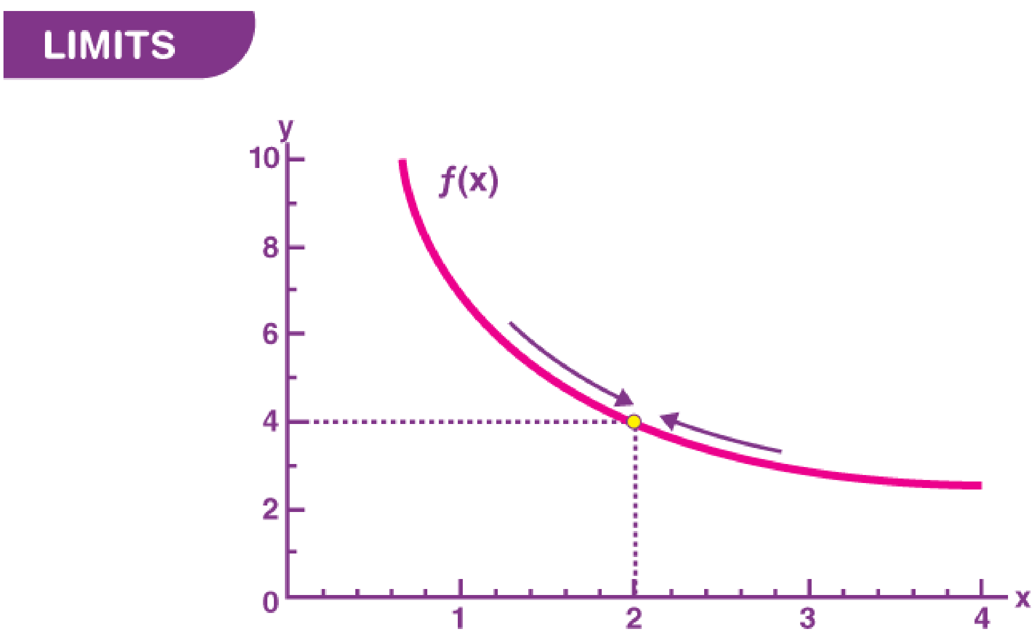
Limits are an essential concept in mathematics and are used to describe the behavior of a function (i.e. $f(x)$, $s(t)$) as it approaches a certain value (i.e. $x = 4$, $t = 10$). We often represent a limit as:

$$\lim_{x \to c} f(x) = L$$

This statement can be read as the following. As the value of variable x approaches the value c, the value from function $f(x)$ nears L.

2.1 Simple Limits

Visually, a limit may look like the following graphic for an arbitrary function:



Example 2 *Simple Limits - Polynomial*
Find the limit of the function $s(t) = 3t^2 - 2t + 1$ as t nears the value 2.

> **Note:**
> When (a) the value c approached is finite and (b) the function $f(x)$ is continuous at the point approached, the value that the function nears will always be the direct calculation of $f(c)$.

Therefore, for this example:

$$\lim_{t \to c} s(t) = \lim_{t \to 2}(3t^2 - 2t + 1) = 3 \cdot 2^2 - 2 \cdot 2 + 1 = 12 - 4 + 1 = 9$$

2.2 End Behavior Limits

Limits become tricky when the approached value is no longer finite. For instance, take $f(x) = 2x + 1$. Though this function's limit can be calculated for any finite number (i.e. $c = 5$, $c = 10^9$, $c = sqrt(\frac{1}{\pi})$), what happens if $c = +\infty$? What if $c = -\infty$?

In these cases, we are observing what occurs at the left and right end behaviors of the function.

Definition 3 ***Right End Behavior:*** *The behavior of the function (i.e. the limit it nears) as the input argument nears $+\infty$, the right end of the function (i.e. $lim_{x \to +\infty} f(x)$)*

Definition 4 ***Left End Behavior:*** *The behavior of the function (i.e. the limit it nears) as the input argument nears $-\infty$, the left end of the function (i.e. $lim_{x \to -\infty} f(x)$)*

> **Note:**
> In limits involving end behaviors of functions, we need to use logic and reasoning to ascertain what the value becomes. There are always four options for what a limit may become:
>
> (a) A finite number
>
> (b) Positive infinity
>
> (c) Negative infinity
>
> (d) Undefined/no limit exists.

Example 3 *Limits Nearing Infinity - Linear Functions*

Find the limit of the function $f(x) = 2x + 1$ as x nears the value $+\infty$. Furthermore, find the limit as t nears the value $-\infty$.

Because the function is nearing $+\infty$, we must consider the right end behavior of the function. The function $f(x) = 2x + 1$ is a straight line, and for straight lines, we can note the following for right end behavior:

(a) If the slope is positive, $f(x)$ nears $+\infty$ as $x \to +\infty$

(b) If the slope is negative, $f(x)$ nears $-\infty$ as $x \to +\infty$

(c) If the slope is zero (i.e. the line $f(x) = k$, where k is a constant), $f(x)$ nears the value k as $x \to +\infty$ (because the function value is independent of the input argument x)

(d) If the slope is undefined, (i.e. the line $x = 5$), the limit is also undefined (the domain as $x \to +\infty$ does not exist)

We can observe the slope, 2, is positive. Therefore the right end behavior of the function $f(x)$ is $+\infty$ as $x \to +\infty$. Therefore,

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (2x + 1) = +\infty$$

Similarly, as $x \to -\infty$, we can note the opposite patterns for left-end behavior for linear functions:

(a) If the slope is positive, $f(x)$ nears $-\infty$ as $x \to -\infty$

(b) If the slope is negative, $f(x)$ nears $+\infty$ as $x \to -\infty$

(c) If the slope is zero (i.e. the line $f(x) = k$, where k is a constant), $f(x)$ nears the value k as $x \to -\infty$ (because the function value is independent of the input argument x)

(d) If the slope is undefined, (i.e. the line $x = 5$), the limit is also undefined (the domain as $x \to -\infty$ does not exist)

We can observe the slope, 2, is positive. Therefore the right end behavior of the function $f(x)$ is $-\infty$ as $x \to -\infty$. Therefore,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (2x + 1) = -\infty$$

Example 4 *Limits Nearing Infinity - Polynomial Functions*

Find the limit of the function $s(t) = 3t^2 - 2t + 1$ as t nears the value $+\infty$. Furthermore, find the limit as t nears the value $-\infty$

We can observe that this parabola is convex. Therefore, as x increases (i.e. nears $+\infty$), the value of $3t^2 - 2t + 1$ also increases so long as the vertex has already been passed. This means that as x gets larger and larger, the values of $s(t) = 3t^2 - 2t + 1$ become larger and larger as well. In fact, the rate at which $s(t)$ grows becomes faster and faster as x increases.

Therefore, we can state that:

$$\lim_{t \to +\infty} s(t) = \lim_{s \to +\infty} (3t^2 - 2t + 1) = +\infty$$

Again, because the parabola is convex, as x decreases in value (i.e. nears $-\infty$), the value of $3t^2 - 2t + 1$ also increases so long as the vertex has already been passed. This means that as x gets smaller and smaller, the values of $s(t) = 3t^2 - 2t + 1$ become larger and larger as well.

Therefore, we can state that:

$$\lim_{t \to -\infty} s(t) = \lim_{s \to -\infty} (3t^2 - 2t + 1) = +\infty$$

Note:

For any function with multiple terms, the overall limit L is a summation of the individual terms limits, $L = \sum_i L_i$, if and only if each limit is a real, finite number.

In other words, if a function can be represented as $f(x) = f_1(x) + f_2(x)$ and both $f_1(x)$ and $f_2(x)$ have finite limits, then $\lim_{x \to c} f(x) = \lim_{x \to c} f_1(x) + \lim_{x \to c} f_2(x)$.

Note:

For any function equal to another function multiplied by a constant (i.e. $f(x) = kg(x)$), the overall limit L is the multiplication of that constant k with the nested function $g(x)$. In other words

$$\lim_{x \to c} f(x) = \lim_{x \to c} kg(x) = k \lim_{x \to c} g(x)$$

We will not prove this rigorously in this lesson.

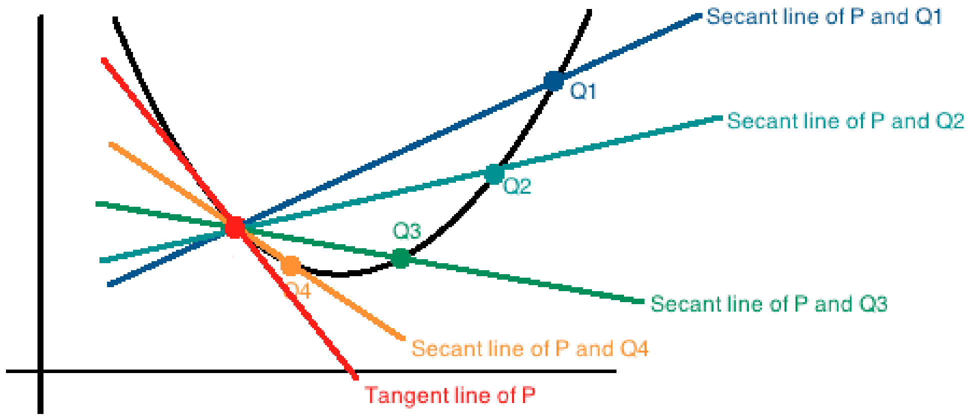
It is important to note that there are many other rules with limits that are not covered here. We encourage you to learn more from additional resources outside of these basic functionalities.

3 Derivatives

In Section 1, we explored how to calculate the average rate of change for any function. In Section 2, we learned the rules for calculating the limit for any arbitrary function. Now, we will combine these two concepts and learn what happens as both points near each other.

3.1 Definitions of a Derivative

Let's take an arbitrary function. If we want to find the average rate of change in an interval, we can use the previous formula defined, Average Rate of Change $= \frac{f(x_1)-f(x_0)}{x_1-x_0}$. Say, for instance, we begin with $x_0 = 0$ and $x_1 = 5$. Now, what if we want to find the rate of change occurring at a time interval closer to $x_1 = 5$? Perhaps instead we can change our interval to be from $x_0 = 4$ and $x_1 = 5$. However, what if this is still too large of an interval?



As the point Q gets closer and closer to P, the secant line joining P and Q becomes more and more like the tangent line of P.

We can keep decreasing the size all we want, but at some point, we come across the issue of $x_0 = 5$ and $x_1 = 5$, and it isn't possible to find an average rate of change using the equation $\frac{f(x_1)-f(x_0)}{x_1-x_0}$ (as we would be dividing by zero). However, what if instead, we take the limit as one point nears the other?

Definition 5 ***Instantaneous Rate of Change:*** *The instantaneous rate of change is a fundamental concept in calculus that measures how a function changes at a specific point. It is also known as a derivative.*

We can define the instantaneous rate of change, also known as the derivative, with the following equation:

$$\text{Instantaneous Rate of Change} = \frac{d}{dx}f(x) = f'(x) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

A derivative of a function is often denoted as $f'(x)$, the "prime" (otherwise known as the apostrophe) denoting that the function $f'(x)$ is the instantaneous rate of change of $f(x)$. Another very common way to write them is $\frac{d}{dx}f(x)$, which can be thought of as the Δ in $\frac{\Delta y}{\Delta x}$ (average rate of change) becoming $\frac{df(x)}{dx}$ when made instantaneous. This can also be the thought of the slope of the line tangent to the point $(x, f(x))$ (see the image above!).

Often in calculus, we prefer to think of the derivative as decreasing the distance between the points rather than having one point approaching another. Therefore, we often reform the derivative like so:

$$\begin{aligned}
f'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x) - f(x + \Delta x)}{x - (x + \Delta x)} && (c = x + \Delta x) \\
&= \lim_{\Delta x \to 0} \frac{f(x) - f(x + \Delta x)}{-\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} && (h = \Delta x)
\end{aligned}$$

Note:

A derivative of a function can be found either from the equation

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

or from the equation

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Example 5 *Definitions of a Derivative - Linear Function*

Find the instantaneous rate of change of $f(x) = 2x + 1$ at $x = 5$.

Here, we will substitute $x = 5$ in as well as the function $f(x) = 2x + 1$ into the previous derivative equation. Our goal is to cancel out the h terms in order to receive a limit that is non-reliant on h, allowing us to directly compute the result.

$$
\begin{aligned}
f'(x=5) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{f(5+h)-f(5)}{h} \\
&= \lim_{h\to 0} \frac{(2(5+h)+1)-(2(5)+1)}{h} \\
&= \lim_{h\to 0} \frac{11+2h-11}{h} \\
&= \lim_{h\to 0} \frac{2h}{h} \\
&= \lim_{h\to 0} 2 \\
&= 2
\end{aligned}
$$

There we have it, our very first derivative! You are now officially performing calculus!

For completeness, let's see what happens when we use the equation $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ instead:

$$
\begin{aligned}
f'(c=5) &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
&= \lim_{x\to 5} \frac{f(x)-f(5)}{x-5} \\
&= \lim_{x\to 5} \frac{(2x+1)-(2(5)+1)}{x-5} \\
&= \lim_{x\to 5} \frac{2x+1-11}{x-5} \\
&= \lim_{x\to 5} \frac{2x-10}{x-5} \\
&= \lim_{x\to 5} \frac{2(x-5)}{x-5} \\
&= \lim_{x\to 5} 2 \\
&= 2
\end{aligned}
$$

Just as expected, we receive the same result as these two equations are analogous to one another!

Note, as we mentioned earlier, any linear function will always have the same average rate of change for any interval. This still holds true for instantaneous rates of change: the resulting solution will always be equal to the slope of the function.

Note:

Any function of the form $f(x) = wx + b$ will always have a derivative equal to $f'(x) = w$

Let's quickly prove this statement!

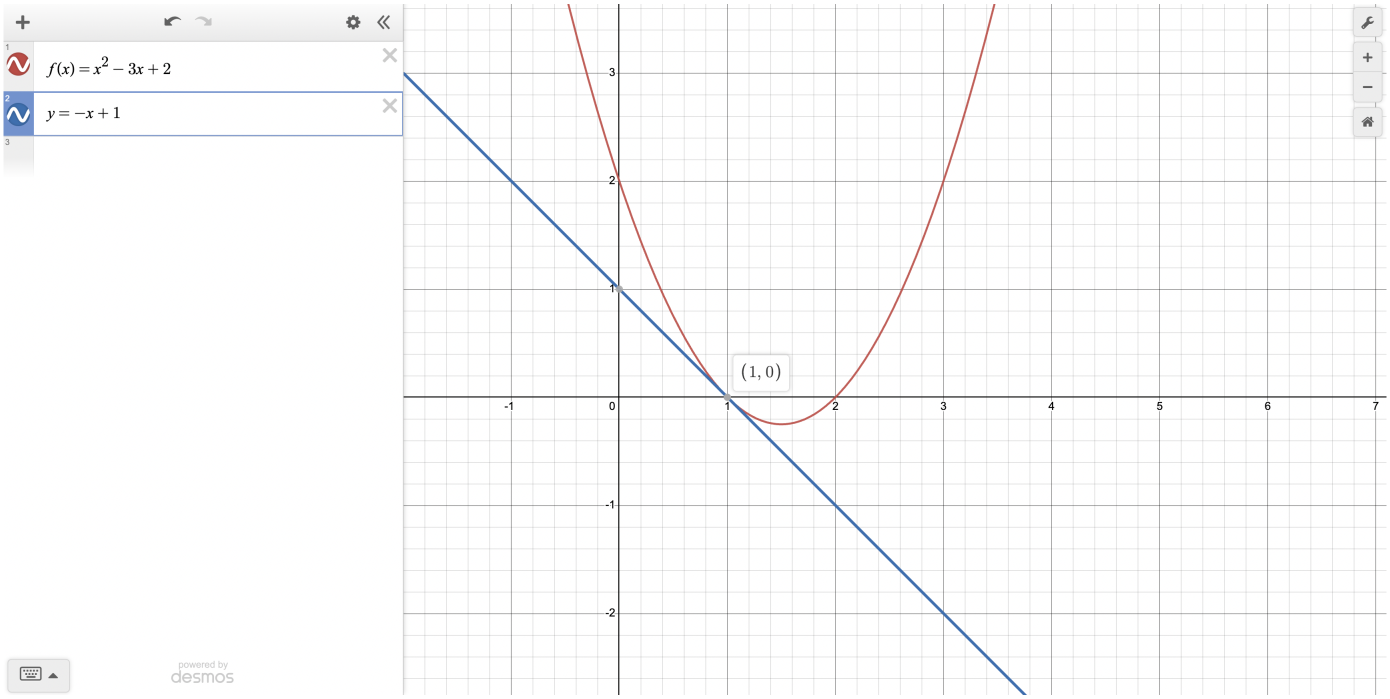
$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{(w(x+h)+b) - (wx+b)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{wx + wh + b - wx - b}{h} \\
f'(x) &= \lim_{h\to 0} \frac{wh}{h} \\
f'(x) &= \lim_{h\to 0} w \\
f'(x) &= w
\end{aligned}$$

Example 6 *Definitions of a Derivative - Quadratic Function*

Find the instantaneous rate of change of $f(x) = x^2 - 3x + 2$ at $x = 1$. Furthermore, graph the function as well as the instantaneous rate of change.

$$\begin{aligned}
f'(x=1) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h\to 0} \frac{((1+h)^2 - 3(1+h) + 2) - ((1)^2 - 3(1) + 2)}{h} \\
&= \lim_{h\to 0} \frac{((1+2h+h^2) - (3+3h) + 2) - 0}{h} \\
&= \lim_{h\to 0} \frac{(1+2h+h^2) - (3+3h) + 2}{h} \\
&= \lim_{h\to 0} \frac{(2h+h^2) - (3h)}{h} \\
&= \lim_{h\to 0} \frac{h^2 - h}{h} \\
&= \lim_{h\to 0} (h-1) \\
&= -1
\end{aligned}$$

Now, let's gain an intuition for how this looks graphically. Let's plot the function $f(x) = x^2 - 3x + 2$ as well as the tangent line created by this derivative. Recall that we can create a line as long as we have a point and a slope (for us, our point is $(x, f(x))$ and our slope is $f'(x = 1) = -1$. Plug this into the equation $y - y_1 = m(x - x_1)$ and we obtain the equation $y = -x + 1$.



We can see that the tangent line indeed resembles the slope of the function at the point $(1, f(1))$! Tangent lines for any function will always look similar, barely skimming the "surface" of the function, parallel to the instantaneous slope.

As this is only an introductory lesson, we will not go into further proofs and problem solutions using the definition of a derivative. However, understand that the definition of a derivative is very powerful. Derivatives can be found for any trigonometric function (i.e. $\sin(x)$, $\arctan(x)$, $\cot(x)$), any natural logarithm (i.e. $\log(x)$, $\ln(x)$), etc. The only derivatives that cannot be taken are when the limits do not exist for a derivative at a particular point. For example, the nonlinear function $y = |x|$ at point $x = 0$ has no derivative (both the left-sided $(f'(x)^+ = 1)$ and right-sided $(f'(x)^- = -1)$ derivatives do not agree).

3.2 Derivative Rules

Derivatives are AWESOME! We can observe the instantaneous rate of change for almost any function using them. The possibilities are endless!

If you could have only one complaint about them, what would it be? If you answered that it's annoying to derive derivatives, then I'd agree with you! It's a pain to have to go through the tedious derivation process for any function. Earlier, we also only derived simple functions: a linear and a quadratic functions' derivatives. What if we had to derive even an even more complex function's derivative?

In this lesson, we will go over some of the various rules for finding derivatives quickly. These are some of the most powerful rules in calculus and make derivatives much more robust and efficient.

Example 7 *Definitions of a Derivative - Constant Rule*

Let's start with the case for an arbitrary constant value. Take $f(x) = k$ for instance. What would be the derivative of this?

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{k - k}{h} \\ &= 0 \end{aligned}$$

Definition 6 ***The Constant Rule:*** *The derivative for any constant valued function of the form $f(x) = k$ is zero.*

$$f'(x) = 0$$

Example 8 *Definitions of a Derivative - Constant Multiple Rule*

Let's examine the derivative of a function times a constant. Take $f(x) = kg(x)$ for instance. What would be the derivative of this?

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{kg(x+h) - kg(x)}{h} \\ &= \lim_{h \to 0} k \frac{g(x+h) - g(x)}{h} \\ &= k \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= kg'(x) \end{aligned}$$

Definition 7 ***The Constant Multiple Rule:*** *The derivative of any function multiplied by a constant (of the form $f(x) = kg(x)$) is the constant times the inner function's derivative.*

$$f'(x) = kg'(x)$$

Example 9 *Definitions of a Derivative - Additive Rule of Differentiation*

Let's examine the derivative of a function plus another function. Take $f(x) = g(x) + m(x)$ for instance. What would be the derivative of this?

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(g(x+h) + m(x+h)) - (g(x) + m(x))}{h} \\
&= \lim_{h\to 0} \frac{(g(x+h) - g(x)) + (m(x+h) - m(x))}{h} \\
&= g'(x) + m'(x)
\end{aligned}$$

Definition 8 ***The Additive Rule of Differentiation:*** *The derivative of two functions added together is the addition of each function's derivative. In other words, if $f(x) = g(x) + m(x)$, then:*

$$f'(x) = g'(x) + m'(x)$$

Example 10 *Definitions of a Derivative - Power Rule*

Let's take the case for an arbitrary polynomial. Take $f(x) = x^n$ for instance, where n is a real, non-zero number. What would be the derivative of this?

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{(x^n + nx^{n-1}h + a_2x^{n-2}h^2 ... + a_{n-1}xh^{n-1} + h^n) - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + a_2x^{n-2}h^2 ... + a_{n-1}xh^{n-1} + h^n}{h} \\
&= \lim_{h\to 0} \frac{h(nx^{n-1} + a_2x^{n-2}h ... + a_{n-1}xh^{n-2} + h^{n-1})}{h} \\
&= \lim_{h\to 0} nx^{n-1} + a_2x^{n-2}h ... + a_{n-1}xh^{n-2} + h^{n-1} \\
&= nx^{n-1} + 0 + 0 + ... \\
&= nx^{n-1}
\end{aligned}$$

Definition 9 ***The Power Rule:*** *The derivative for any polynomial term $f(x) = x^n$ follows the following form:*

$$f'(x) = nx^{n-1}$$

Let's try out these functions and rules!

Example 11 *Basic Derivative Rules - Simple Polynomial*
Find the derivative of $f(x) = 3x^2 - 2x + 1$ for any arbitrary input x.

For the first two terms, we can use the Constant Multiplier Rule, the Power Rule, and the Additive Rule of Differentiation to find that $\frac{d}{dx}(3x^2 - 2x) = 6x - 2$. For the third time, we can use the Constant Rule to state that $\frac{d}{dx}1 = 0$. Therefore, after adding the two derivatives with the Additive Rule of Differentiation, $f'(x) = 6x - 2$.

There are an abundant amount of rules, but in order for us to be concise, we will not give all the proofs necessary for a complete foundation of derivatives. Instead, we will give additional important rules without proof:

Name	Original Function	Derivative
Chain Rule	$f(g(x))$	$f'(g(x))g'(x)$
Product Rule	$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
Quotient Rule	$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
Natural Exponential	e^x	e^x
Arbitrary Exponential	a^x	$a^x \ln(a)$
Natural Logarithm	$ln(x)$	$\frac{1}{x}$
Arbitrary Logarithm	$\log_a(x)$	$\frac{1}{x \ln(a)}$
Sine Function	$\sin(x)$	$\cos(x)$
Cosine Function	$\cos(x)$	$-\sin(x)$
Tangent Function	$\tan(x)$	$\sec^2(x)$

3.3 Critical Points

By now, we have learned what a derivative is and how to find the derivative of arbitrary functions. But, what are they useful for? How are they implemented in the real world?

We will now examine the realm of critical points in order to answer these questions.

Definition 10 ***Critical Points:*** *Critical points are the $(x, f(x))$ coordinate pairs at which the derivative $f'(x)$ of a function $f(x)$ is either equal to zero or is undefined. For any function, at these locations often reside the local (and sometimes global) extremas (maximas and minimas) of a function.*

Let's understand what critical points are through an example problem.

Example 12 *Critical Points - Simple Polynomial*

Find the critical points of the function $f(x) = x^3 - 2x - 1$. Then, plot the function and the corresponding points.

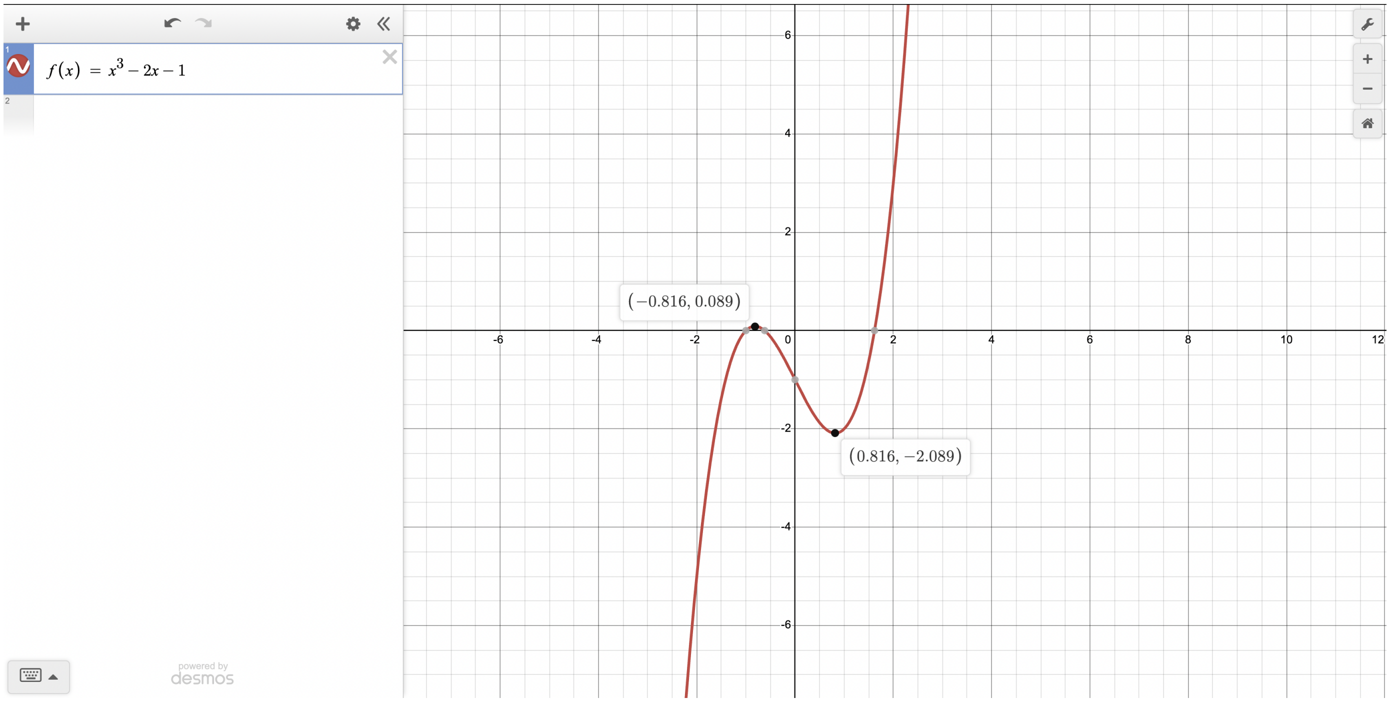
To do this, first we must take the derivative of the function.

$$
\begin{aligned}
f'(x) = 0 &= \frac{d}{dx}(x^3 - 2x - 1) \\
0 &= 3x^2 - 2 \\
x^2 &= \frac{2}{3} \\
x &= \pm\sqrt{\frac{2}{3}}
\end{aligned}
$$

The points respectively on $f(x)$ are:

$$
\begin{aligned}
f(+\sqrt{\frac{2}{3}}) &\approx -2.0887 \\
f(-\sqrt{\frac{2}{3}}) &\approx 0.0887
\end{aligned}
$$

With this information, we can now plot these points on a graph. Note that $\sqrt{(\frac{2}{3})} \approx 0.816$. Therefore, we are plotting the original function, $f(x) = x^3 - 2x - 1$, and the critical points $(0.816, -2.0887)$, $(-0.816, 0.0887)$:



We observe that the critical points lay on the local minima and maxima of the function! Because we are calculating the points at which the slope of the function is equal to zero, these locations are usually points at which a minima or maxima value may reside (there are a few exceptions that we will not cover).

In calculus, this is incredibly useful as we can optimize almost any mathematical system by calculating the critical point of a function. In most real-world problems, we want to maximize a value (i.e. profits, production output, speed) or minimize a value (i.e. costs, time, gas mileage). The ability to do this simply with derivatives, therefore, is incredibly useful.

Example 13 *Critical Points - Sheep Pen Application*

We have 10,000 feet of fence to use in enclosing a rectangular, grassy area. This area will house sheep, and we want to maximize the enclosed area in order to give the sheep as much grass as possible to roam around in. What should the dimensions (height and width) of the fence be?

At first, this problem seems daunting! There are a lot of variables to take into account. However, let's try to form a mathematical statement that represents these parameters first, and then attempt to optimize it afterward!

First, we know the perimeter of the fence can be 10,000 feet long. It is safe to assume that all the fences must be used as a larger perimeter correlates to a larger available area. Therefore, we can state that:

$$\begin{aligned} \text{Perimeter} = P &= 2w + 2h && \text{(Equation for a rectangle's perimeter)} \\ 10000 &= 2w + 2h \end{aligned}$$

Furthermore, we know that the equation for the area of a rectangle is:

$$\text{Area} = A = wh$$

We want to take the derivative of the area formula, as this is the function we want to receive the maxima for. However, we currently have two independent variables in the function! It currently isn't possible!

What if we try rewriting our perimeter formula in order to replace one of the variables? Then, later on, we can solve for that variable after finding the critical point!

Let's rewrite the perimeter formula:

$$\begin{aligned} 10000 &= 2w + 2h \\ 2w &= 10000 - 2h \\ w &= 5000 - h \end{aligned}$$

Now, let's substitute this into the formula for area:

$$\begin{aligned} A &= wh && \text{(Equation for the area of a rectangle)} \\ A &= h(5000 - h) \\ A &= 5000h - h^2 \end{aligned}$$

Now, as we only have one variable, we can take the derivative and find the critical point!

$$\begin{aligned} \frac{d}{dh}A(h) = 0 &= \frac{d}{dh}(5000h - h^2) \\ 0 &= 5000 - 2h \\ h &= 2500 \end{aligned}$$

Now, we plug this back into the perimeter formula to solve for w!

$$w = 5000 - h$$
$$w = 5000 - 2500$$
$$w = 2500$$

We have now solved for the optimal height and width of the fenced area! Coincidentally enough, it turns out that a square-shaped pen best optimizes for a maximum area. Try any other value and you will see that no other shape can yield a larger area than $w = 2500$ ft and $h = 2500$ ft for this given perimeter!

4 Practice Problems

4.1 Rate of Change without Calculus

1. The distance (in kilometers) traveled by car during a 4-hour trip is given by the function $D(t) = 60t + 10$, where t represents the time in hours. Find the average rate of change of the distance with respect to time over the interval [2, 4]

2. Consider the quadratic function $f(x) = -2x^2 + 4x + 3$. Find the average rate of change of f(x) with respect to x over the interval [-1, 3]

3. Consider a horizontal line given by the equation $g(x) = 5$. Find the average rate of change of y with respect to x over the interval [-2, 4]

4.2 Limits

1. Find the limit as x approaches 3 of $f(x) = 2x + 1$.

2. Find the limit as x approaches infinity of $g(x) = -2x^3 - 3000x^2 + 5$

3. Find the limit as x approaches -infinity of $h(x) = 6$

4. Find the limit as x approaches infinity of $j(x) = \frac{5}{x-5}$

4.3 Derivatives

1. Use the definition of a derivative to calculate the derivative of $f(x) = 6x - 3$ (hint: use the formula $\frac{d}{dx}f(x) = f'(x) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ and cancel out like terms)

2. Use the definition of a derivative to calculate the derivative of $g(x) = 3$ (hint: use the formula $\frac{d}{dx}g(x) = g'(x) = \lim_{h \to 0} \frac{g(x+h)-g(x)}{h}$ and cancel out like terms)

3. Use the derivative rules to calculate the derivative of $j(x) = 100x^5 - 3x^2 + 2$

4. Use the derivative rules to calculate the derivative of $k(x) = 9x^8 + 4x^5 - 5x^4 - 580x$