MATH2011/2111 Revision Sheet

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In addition, although the authors have tried their best to include everything essential taught in the course, it was ultimately up to their discretion on whether or not to include results/theorems/definitions etc. Anything that is missing is most definitely a conscious choice made by the authors.

Finally, any and all errors found within this document are most certainly our own. If you have found an error, please contact us via our Facebook page, or give us an email.

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Curves and Surfaces

Curves

Definition: A curve in $\mathbb{R}^n$ is a vector-valued function

$$\mathbf{c} : I \to \mathbb{R}^n$$

where I is an interval on $\mathbb{R}$.

- A **multiple point** is a point through which the curve passes more than once.
- For a curve $\mathbf{c} : [a, b] \to \mathbb{R}^n$, $\mathbf{c}(a)$ and $\mathbf{c}(b)$ are called **end points**.
- A curve is **closed** if its end points are the same point.

Definition: For an interval $I \subset \mathbb{R}$ and curve $\mathbf{c} : I \to \mathbb{R}^n$ with:

$$\mathbf{c}(t) = (c_1(t), c_2(t), \cdots, c_n(t)),$$

- the functions $c_i : I \to \mathbb{R}, i = 1, 2, \cdots, n$ are called the components of $\mathbf{c}$
- $\lim\limits_{t\to a} \mathbf{c}(t) = \left(\lim\limits_{t\to a} c_1(t), \lim\limits_{t\to a} c_2(t), \cdots, \lim\limits_{t\to a} c_n(t)\right)$
- $\dfrac{d\mathbf{c}(t)}{dt} = \dot{\mathbf{c}}(t) = \mathbf{c}'(t) = (c_1'(t), c_2'(t), \cdots, c_n'(t))$

Further,

$$\int_a^b \mathbf{c}(t)dt = \left(\int_a^b c_1(t)dt, \int_a^b c_2(t)dt, \cdots, \int_a^b c_n(t)dt\right)$$

Definition: A curve $\mathbf{c} : I \to \mathbb{R}^n$ is

- **continuous** if its component functions are continuous.
- **simple** if it is continuous and has no multiple points (other that the end points if it's closed).
- **smooth** if its components are differentiable and their derivatives do not simultaneously vanish.
- **piecewise smooth** if it is made up of a finite number of smooth curves.

Surfaces

Surfaces can be described in 4 ways.

1. Graph of a function, $z = f(x, y)$.
2. Implicitly, $f(x, y, z) = c$.
3. Parametrically, $\mathbf{x} = \mathbf{x_0} + \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2}$.
4. Level curves or contour maps can be drawn by equating $f(x, y)$ to constants and sketching each curve.

Analysis

Metrics

A metric is a function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that satisfies the following three properties:

1. Positive definite: $d(\mathbf{x}, \mathbf{y}) \geq 0$, with $d(\mathbf{x}, \mathbf{y}) = 0$ iff $\mathbf{x} = \mathbf{y} = 0$.
2. Symmetric: $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.
3. Triangle Inequality: $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$.

The concept of a metric is important as distance functions are fundamental to defining limits.

Equivalence

Two metrics d_1 and d_2 are called equivalent, if there exist two constants $0 < c < C < \infty$ such that

$$cd_2(\mathbf{x}, \mathbf{y}) \leq d_1(\mathbf{x}, \mathbf{y}) \leq Cd_2(\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

Limits of Sequences

We define a ball around $\mathbf{a} \in \mathbb{R}^n$ with a radius $\epsilon > 0$ as the set of points

$$B(\mathbf{a}, \epsilon) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{a}, x < \epsilon\}.$$

For a sequence $\{\mathbf{x}_i\}$ of points in $\mathbb{R}^n$ we say that $\mathbf{x}$ is the limit of the sequence if and only if

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(\mathbf{x}, \mathbf{x}_n) < \epsilon.$$

That is, there is always a point of the sequence inside any ball centred at $\mathbf{x}$.

Convergence of Sequences

The following conditions are equivalent:

1. A sequence $\mathbf{x}_k$ converges to a limit $\mathbf{x}$
2. The components of $\mathbf{x}_k$ converge to the components of $\mathbf{x}$
3. $d(\mathbf{x}_k, \mathbf{x}) \to 0$ as $k \to \infty$

Open and Closed Sets

Let $\Omega \subset \mathbb{R}^n$. We define:

1. $\mathbf{x}_0 \in \Omega$ is an **interior point** of Ω if there is a ball around $\mathbf{x}_0$ contained in Ω
2. Ω is **open** if every point of Ω is an interior point

3. Ω is **closed** $\leftrightarrow$ its complement is open $\leftrightarrow$ it contains all of its boundary points

4. $\mathbf{x} \in \mathbb{R}^n$ is a **boundary point** of Ω if every ball around $\mathbf{x}_0$ contains both points in Ω and points not in Ω

Note that a set can be BOTH open and closed.

We define $\mathbf{x}_0$ as a **limit point** of Ω if there is a sequence $\{\mathbf{x}_i\} \in \Omega$ with limit $\mathbf{x}_0$ and $\mathbf{x}_i \neq \mathbf{x}_0$

Subsets

We can define the following useful subsets of Ω:

- The **interior** of Ω is the set of all interior points of Ω
- The **boundary** of Ω is the set of all boundary points of Ω, which we denote $\partial\Omega$.
- The **closure** of Ω is $\Omega \cup \partial\Omega$, denoted $\overline{\Omega}$

Limits of Functions

For $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ means that

$$\forall\epsilon > 0, \exists\delta > 0 \text{ such that for } \mathbf{x} \in \Omega$$

$$0 < d(\mathbf{x}, \mathbf{x}_0) < \delta \Rightarrow d(\mathbf{f}(\mathbf{x}), \mathbf{b}) < \epsilon$$

Now, for $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ with $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = a$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = b$, we have

- $\lim_{\mathbf{x}\to\mathbf{x}_0}(f + g)(\mathbf{x}) = a + b$.
- $\lim_{\mathbf{x}\to\mathbf{x}_0}(fg)(\mathbf{x}) = ab$
- $\lim_{\mathbf{x}\to\mathbf{x}_0}(f/g)(\mathbf{x}) = a/b$ provided $b \neq 0$

Continuity

Suppose that $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{x}_0 \in \Omega$, then then following are equivalent:

1. $\mathbf{f}$ is continuous at $\mathbf{x}_0 \in \Omega$
2. If $\mathbf{x} \in B(\mathbf{x}_0, \delta) \Rightarrow \mathbf{f}(\mathbf{x}) \in B(\mathbf{f}(\mathbf{x}_0), \epsilon)$.
3. $\forall$ sequences $\mathbf{x}_k$ in Ω with limit $\mathbf{x}_0$, $\mathbf{f}(\mathbf{x}_k)$ converges to $\mathbf{f}(\mathbf{x}_0)$.
4. $\mathbf{f}(\mathbf{x}_0)$ is interior point of $\mathbf{f}(\Omega) \Rightarrow \mathbf{x}_0$ is an interior point of Ω.

Additionally, if $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, then the following are also equivalent:

1. $\mathbf{f}$ is continuous on Ω
2. U is open in $\mathbb{R}^m \Rightarrow \mathbf{f}^{-1}(U)$ is open in $\mathbb{R}^n$.

Very Useful Continuity Theorems

- If $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is an elementary function, then $\mathbf{f}$ is continuous on Ω.
- Additionally, $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous on Ω iff its component functions are also continuous.
- Now, for two functions continuous on Ω, $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ and $g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, then $f + g$, fg and f/g are all continuous (assuming that the domain of f/g excludes points where g($\mathbf{x}$)=0).

Compact and Connected Sets

Definition: We define a set $\Omega \subset \mathbb{R}^n$ as bounded if there is some M such that $d(\mathbf{x}, \mathbf{0}) \leq M$ for all $\mathbf{x} \in \Omega$.

The **Bolzano-Weierstrass Theorem**: For $\Omega \subset \mathbb{R}^n$, the following are equivalent:

- Ω is closed and bounded.
- Every sequence in Ω has a subsequence that converges to an element of Ω.

A set Ω is **compact** if it satisfies either property from the Bolzano-Weierstrass theorem (though we usually use the first one)

A **continuous path** between $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$ is a function $\phi : [0, 1] \to \mathbb{R}^n$ such that ϕ is continuous and $\phi(0) = \mathbf{x}$, $\phi(1) = \mathbf{y}$.

A set $\Omega \subset \mathbb{R}^n$ is said to be **path connected** if for every $\mathbf{x}, \mathbf{y} \in \Omega$, there is a continuous path between them that lies entirely in Ω.

The Big Theorems

Let $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. Then

- $K \subset \Omega$ and K is compact $\Rightarrow \mathbf{f}(K)$ is compact.
- $B \subset \Omega$ and B is path connected $\Rightarrow \mathbf{f}(B)$ is path connected.

That is, continuous functions preserve compactness and path connectedness.

Differentiable Functions

Affine Approximations

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is affine, then there is $\mathbf{y}_0 \in \mathbb{R}^m$ and a linear map (matrix) $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{x}) = \mathbf{y}_0 + \mathbf{L}(\mathbf{x})$$

A good affine approximation for $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ at $\mathbf{a} \in \mathbb{R}^n$ is given by

$$\mathbf{T}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + \mathbf{D}(\mathbf{a})(\mathbf{x} - \mathbf{a}),$$

Where $\mathbf{D}(\mathbf{a})$ is the derivative of $\mathbf{f}$ at $\mathbf{a}$.

Derivatives

A function $\mathbf{f} : \Omega \in \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $\mathbf{a} \in \Omega$ if there is a linear map $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x \to \mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - \mathbf{L}(\mathbf{x} - \mathbf{a})\|}{\|\mathbf{x} - \mathbf{a}\|} = 0$$

The matrix of the map $\mathbf{L}$ is the *derivative* of $\mathbf{f}$ at $\mathbf{a}$, and is denoted $D_{\mathbf{a}}\mathbf{f}$.

Partial Derivatives

Consider some $f(x, y)$. If we fix the value of y, we consider the rate of change of f only as x changes. This is called the *partial derivative* of $f(x, y)$ with respect to x. Formally, it is given by

$$\frac{\partial}{\partial x} f(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Computing partial derivatives is very similar to finding single variable derivatives.

We can extend this to more general multivariable functions. Let $\mathbf{a} \in \Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a function with coordinates x_i and a standard basis $\mathbf{e}_i$, where $i = 1, 2, ..., n$. The partial derivative of f with respect to x_i is defined as:

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{e}_i) - f(\mathbf{a})}{h}$$

if the limit exists.

Clariaut's Theorem

If $f, \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}$ all exist and are continuous on an open set around $\mathbf{a}$, then

$$\frac{\partial^2 f}{\partial x \partial y}(\mathbf{a}) = \frac{\partial^2 f}{\partial y \partial x}(\mathbf{a}).$$

That is, the partial derivatives commute.

Jacobian Matrix

If all the partial derivatives of $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ exist at $\mathbf{a} \in \Omega$, then the **Jacobian** matrix of $\mathbf{f}$ at $\mathbf{a}$ is

$$J_{\mathbf{a}}\mathbf{f} = \begin{bmatrix} \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1}(\mathbf{a}) & \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2}(\mathbf{a}) & \cdots & \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_n}(\mathbf{a}) \\ \frac{\partial \mathbf{f}_2}{\partial \mathbf{x}_1}(\mathbf{a}) & \frac{\partial \mathbf{f}_2}{\partial \mathbf{x}_2}(\mathbf{a}) & \cdots & \frac{\partial \mathbf{f}_2}{\partial \mathbf{x}_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{f}_m}{\partial \mathbf{x}_1}(\mathbf{a}) & \frac{\partial \mathbf{f}_m}{\partial \mathbf{x}_2}(\mathbf{a}) & \cdots & \frac{\partial \mathbf{f}_m}{\partial \mathbf{x}_n}(\mathbf{a}) \end{bmatrix}$$

If $\mathbf{f}$ is differentiable at $\mathbf{a}$, then all the partial derivatives of the components of $\mathbf{f}$ exist at $\mathbf{a}$, and $D_{\mathbf{a}}\mathbf{f} = J_{\mathbf{a}}\mathbf{f}$.

Note that the existence of the Jacobian matrix at a point does not imply that the function is differentiable at that point.

- To prove differentiability, the definition must be used.

Differentiability and Continuity

- Suppose $\Omega \in \mathbb{R}^n$ is open and $\mathbf{f}$ is differentiable on Ω. Then $\mathbf{f}$ is continuous on Ω.
- Additionally, if all the partial derivative sof f exist and are continuous, then f is differentiable. Thus, we obtain:

  Continuous Partials $\Rightarrow$ Diffferentiable $\Rightarrow$ Continuous

Gradient

The Jacobian of $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a vector we call the *gradient* of f. That is,

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right)$$

Using the gradient, the best affine approximation for $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{a}$ is $T(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$.

Chain Rule

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : \Omega' \subset \mathbb{R}^m \to \mathbb{R}^p$ with $f(\Omega) \subset \Omega'$. Suppose f, g are differentiable at $\mathbf{a}$, then so is $g \circ f : \Omega \to \mathbb{R}^p$, and the chain rule is that

$$D(g \circ f)(\mathbf{a}) = Dg(f(\mathbf{a}))Df(\mathbf{a})$$

Directional Derivative

The directional derivative measures the rate of change in an arbitrary, non-basis direction.

For $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ the directional derivative in the direction of the unit vector $\hat{\mathbf{u}}$ at $\mathbf{a} \in \Omega$ is:

$$D_{\hat{\mathbf{u}}} f(\mathbf{a}) = f'_{\hat{\mathbf{u}}}(\mathbf{a}) = \lim_{t \to 0} \frac{f(\mathbf{a} + t\hat{\mathbf{u}}) - f(\mathbf{a})}{t}$$

If $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$ then $D_{\hat{\mathbf{u}}} f(\mathbf{a})$ exists and is equal to:

$$\frac{\partial f}{\partial \mathbf{u}} = D_{\hat{\mathbf{u}}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \hat{\mathbf{u}}$$

That is if $\nabla f(\mathbf{a})$ exists, the directional derivative is just a dot product between the gradient of f at $\mathbf{a}$ and the unit vector in some direction.

This relation gives rise to some geometric properties about the gradient:

1. The maximum rate of change of f at $\mathbf{a}$ occurs in the direction of $\nabla f(\mathbf{a})$
2. The minimum rate of change of f at $\mathbf{a}$ occurs in the direction of $-\nabla f(\mathbf{a})$
3. f does not change in directions perpendicular to $\nabla f(\mathbf{a})$, which are tangents to the level curves of f

Tangent Planes

- For a surface in $\mathbb{R}^3$ defined by $\varphi(x, y, z) = c$, where $\varphi : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, the gradient $\nabla\varphi$ is normal to the surface.
- Thus a tangent plane to the surface at the point $\mathbf{a}$ is given by $\nabla\varphi(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0$ (point-normal form).
- This is equivalent to the plane given by the best affine approximation, $T(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$

Taylor Series

Definition: A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is C^r or **of class** r on the open set Ω if every partial derivative of f of order less than or equal to r exists and is continuous.

Taylor polynomials can be extended to multivariable functions where:

$$f(\mathbf{x}) = P_{r,\mathbf{a}}(\mathbf{x}) + R_{r,\mathbf{a}}(\mathbf{x})$$

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be C^2. The first order Taylor polynomial around $\mathbf{a}$ is given by:

$$P_{1,\mathbf{a}}(\mathbf{x}) = f(\mathbf{a}) + \nabla f \cdot (\mathbf{x} - \mathbf{a})$$

Hessian Matrix

For a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, define the **Hessian matrix** at $\mathbf{a}$ as

$$Hf(\mathbf{a}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{a}) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{a}) & \frac{\partial f}{\partial x_2 \partial x_n}(\mathbf{a}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{a}) \end{bmatrix}$$

The remainder for the first order Taylor polynomial is given by

$$R_{r,\mathbf{a}}(\mathbf{x}) = \frac{1}{2}((\mathbf{x} - \mathbf{a}) \cdot Hf(\mathbf{z})(\mathbf{x} - \mathbf{a}))$$

where $\mathbf{z}$ is a point on the line segment joining $\mathbf{x}$ and $\mathbf{a}$.

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be C^2. The second order Taylor polynomial around $\mathbf{a}$ is given by:

$$P_{2,\mathbf{a}}(\mathbf{x}) = f(\mathbf{a}) + \nabla f \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}\left((\mathbf{x} - \mathbf{a}) \cdot Hf(\mathbf{a})(\mathbf{x} - \mathbf{a})\right)$$

Taylor's Theorem in $\mathbb{R}^n$

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be C^r on the open set Ω. Let $\mathbf{a} \in \Omega$ be such that the line segment joining a and x lies in Ω. Then,

$$f(\mathbf{x}) = P_{r,\mathbf{a}}(\mathbf{x}) + R_{r,\mathbf{a}}(\mathbf{x})$$

where

$$P_{r,\mathbf{a}}(\mathbf{x}) = f(\mathbf{a}) + \nabla f \cdot (\mathbf{x} - \mathbf{a}) + \cdots + \\ \frac{1}{r!} \sum_{i_1,\ldots,i_r=1}^{n} \frac{\partial^r f}{\partial x_{i_1} \cdots \partial x_{i_r}}(\mathbf{a})(x_{i_1} - a_{i_1}) \cdots (x_{i_r} - a_{i_r})$$

Classification of Stationary Points

Definitions

Let $\mathbf{a} \in \Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a function. Then $\mathbf{a}$ is a:

- $\mathbf{a}$ is a *stationary point* if $\nabla f(\mathbf{a}) = \mathbf{0}$ and f is differentiable at $\mathbf{a}$
- $\mathbf{a}$ is a *global maximum* if $f(\mathbf{a}) \geq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega$
- $\mathbf{a}$ is *global minimum* if $f(\mathbf{a}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega$
- $\mathbf{a}$ is a *local maximum* if there is an open set A containing $\mathbf{a}$ such that $f(\mathbf{a}) \geq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega \cap A$
- $\mathbf{a}$ is a *local minimum* if there is an open set A containing $\mathbf{a}$ such that $f(\mathbf{a}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega \cap A$
- $\mathbf{a}$ is a *saddle point* if it is a stationary point of f but is neither a local maximum nor minimum of f.

A Hessian Matrix is used to classify stationary points. An $n \times n$ matrix H is

- positive definite iff all eigenvalues are > 0,
- positive semi-definite iff all eigenvalues are ≥ 0,
- negative definite iff all eigenvalues are < 0,
- negative semi-definite iff all eigenvalues are ≤ 0.

These characteristics can tell us if a point is a local maximum or minimum and vice versa:

- $H(f, \mathbf{a})$ is +ve definite $\Rightarrow f$ has a local min at $\mathbf{a}$,
- $H(f, \mathbf{a})$ is -ve definite $\Rightarrow f$ has a local max at $\mathbf{a}$,
- f has a local min at $\mathbf{a} \Rightarrow H(f, \mathbf{a})$ is +ve semi-definite,
- f has a local max at $\mathbf{a} \Rightarrow H(f, \mathbf{a})$ is -ve semi-definite.

Sylvester's Criterion

Let Δ_k be the determinant of the upper $k \times k$ submatrix of H. Then H is

- positive definite if and only if $\Delta_k > 0$ for all k.
- positive semidefinite if $\Delta_k \geq 0$ for all k.
- negative definite if and only if $\Delta_k < 0$ for all odd k and $\Delta_k > 0$ for all even k.
- negative semidefinite if $\Delta_k \leq 0$ for all odd k and $\Delta_k \geq$ for all even k.

Finding Minimum and Maximum Values

Suppose $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Minimum and maximum values can only occur at stationary points, on the boundary or at places where f is not differentiable.

To find the minimum and maximum values of a function f at a given region:

1. Find the stationary points by letting $\nabla f = \mathbf{0}$ and substituting them back into f.
2. Find boundary points by first coming up with a parametric for the boundary, subbing this parametric in and differentiating to find its minimum and maximum values.
3. Look at what points are not able to be differentiated in the given region and find the values at these points.
4. Compare the points and pick the lowest and/or highest values.

Lagrange Multipliers

Theorem: Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are differentiable. Let there be a constraint $g(x) = c$ on f.

If $g(\mathbf{a}) = c$ and $\mathbf{a}$ is a local max or min on f within this constraint, then there exists a constant λ such that

$$\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a})$$

Note: $\nabla g(a) \neq 0$.

Steps to finding a local max or min on f with the constraint $g(\mathbf{x}) = c$:

1. Find ∇f and ∇g and solve the equation $\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$
2. Use each component to create equations for x, y, z in terms of x, y, z
3. Substitute these equations back into f to get points where $\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x})$
4. Evaluate f at these points and determine if it is a local maximum or minimum

Suppose $f, g_1, g_2, ..., g_n$ are differentiable functions with domain $\mathbb{R}^n$ and codomain $\mathbb{R}$. Suppose there are multiple constraints

$$g_1(\mathbf{x}) = c_1, g_2(\mathbf{x}) = c_2, ..., g_n(\mathbf{x}) = c_n$$

If there is a local minimum or maximum, $\mathbf{a}$ for f within these constraints, then there exists constants $\lambda_1, \lambda_2, ..., \lambda_n$ such that

$$\nabla f(\mathbf{a}) = \lambda_1 \nabla g_1(\mathbf{a}) + \lambda_2 \nabla g_2(\mathbf{a}) + ... + \lambda_n \nabla g_n(\mathbf{a})$$

Inverse Function Theorem

Let $\Omega \subset \mathbb{R}^n$ be open and $\mathbf{f} : \Omega \to \mathbb{R}^n$ be C^1. Let $\mathbf{a} \in \Omega$ and U be an open set around $\mathbf{a}$.

If the matrix $Df(\mathbf{a})$ is invertible, then f is invertible on some U such that a function

$$\mathbf{f}^{-1} : \mathbf{f}(U) \to U$$

exists. Further, $\mathbf{f}^{-1}$ will also be C^1 and for $\mathbf{x} \in U$,

$$D_{\mathbf{f}(\mathbf{x})}\mathbf{f}^{-1} = (D_{\mathbf{x}}\mathbf{f})^{-1}$$

This is somewhat analogous to the single variable inverse function theorem.

Affine Approximation

$\mathbf{f}^{-1}$ has a good affine approximation given at $f(\mathbf{a})$ given by:

$$\mathbf{f}^{-1}(\mathbf{x}) \approx \mathbf{a} + (D_{\mathbf{a}}\mathbf{f})^{-1}(\mathbf{x} - \mathbf{f}(\mathbf{a}))$$

Implicit Function Theorem

The implicit function theorem is used to locally express the graph of a surface as the graph of a function. For example, expressing a surface $F(x, y, z) = 0$ as a function $z = f(x, y)$.

Supose $\mathbf{x} \in \mathbb{R}^m$ are known variables and $\mathbf{u} \in \mathbb{R}^n$ are unknown, and we wish to express $\mathbf{u}$ in terms of $\mathbf{x}$. Let $\mathbf{g} : \mathbb{R}^{m+n} \to \mathbb{R}^n$ be a function such that $\mathbf{g}(\mathbf{x}, \mathbf{u}) = 0$ is a surface containing $(\mathbf{x}_0, \mathbf{u}_0)$.

Note that this condenses the individual equations

$$\begin{aligned} g_1(x_1, ..., x_m, u_1, ..., u_n) &= 0 \\ g_2(x_1, ..., x_m, u_1, ..., u_n) &= 0 \\ &\vdots \\ g_n(x_1, ..., x_m, u_1, ..., u_n) &= 0 \end{aligned}$$

where $\mathbf{g}(\mathbf{x}, \mathbf{u}) = (g_1(\mathbf{x}, \mathbf{u}), g_2(\mathbf{x}, \mathbf{u}), ..., g_n(\mathbf{x}, \mathbf{u}))$, which is the system of n equations necessary.

The goal to find some function $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$ with $\mathbf{g}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = 0$. The statement of the *implicit function theorem* is that:

If $B(\mathbf{x}_0, \mathbf{u}_0)$ is an invertible matrix, then there is an open set V around $\mathbf{x_0}$ on which $\mathbf{u}$ is defined implicitly as a function of $\mathbf{x}$. That is, there exists a continuously differentiable function $\mathbf{f}$ such that for all $\mathbf{x} \in V$, $\mathbf{g}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = 0$. Note that B is defined as,

$$D\mathbf{g} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_m} & | & \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_n} \\ \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_m} & | & \frac{\partial g_n}{\partial u_1} & \cdots & \frac{\partial g_n}{\partial u_n} \end{bmatrix} = [A|B]$$

Pratically, to find $D\mathbf{f}$ in terms of $D\mathbf{g}$, the chain rule is used, giving:

$$D\mathbf{f}(\mathbf{x}) = -B(\mathbf{h}(\mathbf{x}))^{-1} A(\mathbf{h}(\mathbf{x}))$$

where $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^{m+n}$ is defined by:

$$\begin{aligned} \mathbf{h}(\mathbf{x}) &= (\mathbf{x}, \mathbf{f}(\mathbf{x})) \\ &= (x_1, ..., x_m, f_1(x_1, ..., x_m), ..., f_2(x_1, ..., x_m)) \end{aligned}$$

Integration

Riemann Integral

Riemann Sums

The Riemann sums can be extended into multiple dimensions. Consider $f : R \to \mathbb{R}$ where R is the rectangle $[a, b] \times [c, d]$. Let

$$\begin{aligned} \mathcal{P}_1 &= \{a = x_-, x_1, ..., x_n = b\} \\ \mathcal{P}_2 &= \{c = y_-, y_1, ..., y_m = d\} \end{aligned}$$

be partitions of [a,b] and [c,d] respectively. The upper sum is defined by

$$\overline{\mathcal{S}}_{\mathcal{P}_1, \mathcal{P}_2} = \sum_{j,k} \overline{f}_{j,k} \Delta x_j \Delta y_k$$

The lower sum is similarly defined by

$$\underline{\mathcal{S}}_{\mathcal{P}_1, \mathcal{P}_2} = \sum_{j,k} \underline{f}_{j,k} \Delta x_j \Delta y_k$$

where the sum is over all pairs (j, k) for $1 \leq j \leq n$ and $1 \leq k \leq m$, and $\overline{f}_{j,k}$ represents the supremum of f over the rectangle $[x_{j-1}, x_j] \times [y_{k-1}, y_k]$.

Definition

If there exists a unique number $I \in \mathbb{R}$ such that

$$\underline{\mathcal{S}}_{\mathcal{P}_1, \mathcal{P}_2} \leq I \leq \overline{\mathcal{S}}_{\mathcal{P}_1, \mathcal{P}_2}$$

for every pair of partitions $\underline{\mathcal{S}}_{\mathcal{P}_1, \mathcal{P}_2}$, $\overline{\mathcal{S}}_{\mathcal{P}_1, \mathcal{P}_2}$, then f is *Reimann integrable* and

$$I = \iint_R f = \iint_R f(x, y) dA$$

where I is the **Riemann integral**.

Properties of Riemann Integral

Let f, g be Riemann integrable on R.

1. Linearity: $\displaystyle\iint_R \alpha f + \beta g = \alpha \iint_R f + \beta \iint_R g$
2. Positivity or monoticity: Where $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in R$ then $\displaystyle\iint_R f \leq \iint_R g$
3. $\displaystyle\left| \iint_R \right| f \leq \iint_R |f|$
4. Where $R = R_1 \cup R_2$ and interior $R_1 \cap$ interior $R_2 = \emptyset$ then

$$\iint_R f = \iint_{R_1} f + \iint_{R_2} f$$

Fubini's Theorem

Fubini's Theorem

On a rectangular domain $R = [a,b] \times [c,d]$ with a continuous function $f : R \to \mathbb{R}$,

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy = \iint_R f$$

Integrability of Bounded Functions:

- If f is bounded on the rectangle R and the set of points where f is discontinuous lies on a finite union of graphs of continuous functions, then f is integrable on R.

Extension of Fubini's theorem:

- If $f : R \to \mathbb{R}$ is a bounded function on a rectangular domain $R = [a,b] \times [c,d]$ with the discontinuities confined to finite union of graphs of continuous functions. Then, if the integral $\int_c^d f(x,y)dy$ exists for each $x \in [a,b]$, then

  $$\iint_R f = \int_a^b \left(\int_c^d f(x,y)dy \right) dx$$

  If the integral $\int_a^b f(x,y)dx$ exists for each $y \in [c,d]$, then

  $$\iint_R f = \int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

Elementary Regions

Elementary regions are defined as either y-simple or x-simple.

- A region D_1 is y-simple if there exists continuous functions $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ such that

  $$D_1 = \{(x,y) : x \in [a,b] \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}$$

- A region D_2 is x-simple if there exists continuous functions $\psi_1 \psi_2 : [c,d] \to \mathbb{R}$ such that

  $$D_2 = \{(x,y) : y \in [c,d] \text{ and } \psi_1(y) \leq x \leq \psi_2(y)\}$$

Multiple Integrals

Let D_1 be a y-simple region contained within the rectangle $R = [a,b] \times [c,d]$ which is bounded by the curves $y = \phi_1(x)$ and $y = \phi_2(x)$, and let $f : D_1 \to \mathbb{R}$ be function. Then

$$\iint_{D_1} f = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dydx$$

If D_2 is an x-simple region contained in R bounded by $x = \psi_1(y)$ and $x = \psi_2(y)$, then

$$\iint_{D_2} f = \int_c^d \int_{psi_1(y)}^{\psi_2(y)} f(x,y)dxdy$$

Leibniz' Rule

Uniform Continuity

- A function $f : \mathit{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^m$ is **uniformly continuous** on $\mathit{\Omega}$ if for all $\mathbf{x}, \mathbf{y} \in \mathit{\Omega}$ and for any $\varepsilon \in \mathbb{R}^+$ there exists δ such that

  $$d(\mathbf{x}, \mathbf{y}) < \delta \implies d(f(\mathbf{x}), f(\mathbf{y})) < \varepsilon$$

- A continuous function on a compact set is uniformly continuous on that set.

Differentiation under the Integral Sign

For a function $f : \mathit{\Omega} \subset \mathbb{R}^2 \to \mathbb{R}$ that is continuous on $\mathit{\Omega}$ and $\dfrac{\partial f}{\partial x}$ is uniformly continuous on $\mathit{\Omega}$. If

$$F(x) = \int_a^b f(x,y)dy$$

Then

$$\begin{aligned} F'(x) &= \frac{d}{dx} \int_a^b f(x,y)dy \\ &= \int_a^b \frac{\partial f}{\partial x}(x,y)dy \end{aligned}$$

Variable Limit Case

In the case of variable limits, such as

$$\frac{d}{dt} \int_{u(t)}^{v(t)} f(x,t)dx$$

Define $w(t) = t$, and define $F(u,v,w) = \int_u^v f(x,w)dx$. Then

$$\begin{aligned} \frac{d}{dt} \int_{u(t)}^{v(t)} f(x,t)dx &= \frac{d}{dt} F(u,v,w) \\ &= -f(u,t)\frac{du}{dt} + f(v,t)\frac{dv}{dt} + \int_u^v \frac{\partial f}{\partial t}(x,t)dx \end{aligned}$$

by the chain rule and the usual Leibniz rule.

Change of Variables

Let $\mathit{\Omega} \subset \mathbb{R}^n$ and $F : \mathit{\Omega} \to \mathbb{R}^n$ be one to one and C^1 such that $\det(JF(\mathbf{x})) \neq 0$ for all $\mathbf{x} \in \mathit{\Omega}$.

Let $\mathit{\Omega}' = F(\mathit{\Omega})$ and $f : \mathit{\Omega}' \to \mathbb{R}^m$. Then,

$$\int_{\mathit{\Omega}'} f = \int_{\mathit{\Omega}} (f \circ F)|\det(JF)|$$

An alternative statement with two variables is for $x(u,v)$ and $y(u,v)$ then

$$\int_{\mathit{\Omega}'} f(x,y)dxdy = \int_{\mathit{\Omega}} f(x(u,v), v(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det(JF) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Polar Coordinates

With *cylindrical polar coordinates*, the transformations

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

are made. In this case, $\det(JF) = r$.

With *spherical polar coordinates*, the transformations

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

are made. In this case, $\det(JF) = \rho^2\sin\phi$.

Fourier Series

Fourier coefficients

Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function that is square integrable, i.e. $\int_{-L}^{L} f(x)^2 \; dx < \infty$.

Then the Fourier series representation of $f(x)$ is a

$$S_f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right]$$

where

- $a_k = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right) dx$ for $k = 0, 1, 2, \ldots$
- $b_k = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right) dx$ for $k = 1, 2, \ldots$

Odd and even functions

If f is odd, then the Fourier series becomes

$$S_f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right).$$

If f is even, then the Fourier series becomes

$$S_f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right).$$

Inner Products and Norms

Inner Products

Let V be a real vector space. An inner product on V fulfills the following conditions

1. Positive definite:
   $\langle f, f\rangle \geq 0$ and $\langle f, f\rangle = 0$ if and only if $f = 0$
2. Linear with respect to the first argument:
   $\langle \lambda f + \mu g, h\rangle = \lambda\langle f, h\rangle + \mu\langle g, h\rangle$
3. Symmetric: $\langle g, f\rangle = \langle f, g\rangle$

for all functions $f, g, h \in V$ and all constants $\lambda, \mu \in \mathbb{R}$. Conditions 2 and 3 imply linearity w.r.t to the second argument.

Example: The vector space $C[a, b]$ consisting of all continuous functions defined on the interval $[a, b]$ admits the inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x) \; dx.$$

We say f and g are orthogonal in $[a, b]$ if $\langle f, g\rangle = 0$.

Norms

A norm on V fulfills the following conditions

1. Positive definite:
   $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f = 0$
2. $\|\lambda f\| = |\lambda| \, \|f\|$
3. Triangle inequality: $\|f + g\| \leq \|f\| + \|g\|$

for all functions $f, g \in V$ and constant $\lambda \in \mathbb{R}$.

Examples: The vector space $C[a, b]$ admits the following norms:

1. L^2-norm: $\|f\|_2 = \sqrt{\int_a^b f(x)^2 \, dx}$
2. Max-norm: $\|f\|_\infty = \max\limits_{a \leq x \leq b} |f(x)|$

The vector space $B[a, b]$ consisting of all bounded functions defined on the interval $[a, b]$ admits the following norm

1. $\|f\| = \sup\limits_{a \leq x \leq b} |f(x)|$

Theorem: Every inner product on a vector space V induces a norm given by

$$\|f\| = \sqrt{\langle f, f \rangle},$$

and the Cauchy-Schwartz inequality holds

$$|\langle f, g \rangle| \leq \|f\| \|g\| \text{ for all } f, g \in V.$$

Pointwise convergence of Fourier Series

Piecewise continuous functions

Consider a funciton $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$. Suppose that the one-sided limits

$$f(c^+) = \lim_{x \to c^+} f(x)$$

and

$$f(c^-) = \lim_{x \to c^-} f(x)$$

exist.

- If $f(c^+) = f(c^-)$, then f is continuous at c.
- If $f(c^+) = f(c^-) \neq f(c)$ or if $f(c)$ is undefined, then f has a removable discontinuity at c. It this case, we can redefine $f(c)$ to make f continuous at c.
- If $f(c^+) \neq f(c^-)$, then f has a jump discontinuity at c.

Definition: A function f is piecewise continuous on $[a, b]$ if and only if

1. for each $x \in [a, b)$, $f(x^+)$ exists
2. for each $x \in (a, b]$, $f(x^-)$ exists
3. f is continuous on (a, b) except at at most a finite number of points

Theorem: Let f be a piecewise continuous function on a closed bounded interval $[a, b]$. Then $\int_a^b f(x) dx$ exists.

Piecewise differentiable functions

Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$. We write

$$D^+ f(c) = \lim_{h \to 0^+} \frac{f(c + h) - f(c^+)}{h}$$

and

$$D^- f(c) = \lim_{h \to 0^-} \frac{f(c + h) - f(c^-)}{h}$$

if these one-sided limits exist.

- A function f is differentiable at c if and only if $f(c^+) = f(c) = f(c^-)$ and $D^+ f(c) = D^- f(c)$.

Definition: A function f is piecewise differentiable on $[a, b]$ if and only if

1. for each $x \in [a, b)$, $D^+ f(x)$ exists
2. for each $x \in (a, b]$, $D^- f(x)$ exists
3. f is differentiable on (a, b), except at at most a finite number of points

Theorem: Let $c \in \mathbb{R}$, and suppose that a function $f : \mathbb{R} \to \mathbb{R}$ has the following properties:

1. f is 2π-periodic
2. f is piecewise continuous on $[-\pi, \pi]$
3. $D^+ f(c)$ and $D^- f(c)$ exist

If f is continuous at c, then the Fourier series of f converges at c, and its value agrees with f at c, that is $S_f(c) = f(c)$.
Alternatively, if f has a jump/removable discontinuity at c, then

$$S_f(c) = \frac{1}{2} \left[f(c^+) + f(c^-) \right]$$

Convergence of sequences of functions

Pointwise convergence

Definition: Let $f_k : \mathbb{R} \to \mathbb{R}$. We say that f_k converges to f on $[a, b]$ pointwisely if and only if for every $x \in [a, b]$, $f_k(x) \to f(x)$ as $k \to \infty$.
In $\delta - \epsilon$ terms: For every $x \in [a, b], \epsilon > 0$, there exists K such that

$$|f_K(k) - f(x)| \leq \epsilon \text{ for all } k \leq K$$

Uniform convergence

Definition: Let $f_k : \mathbb{R} \to \mathbb{R}$. We say that f_k converges to f on $[a, b]$ uniformly if and only if for every $\epsilon > 0$, there exists a K such that

$$\sup_{x\in[a,b]} |f_k(x) - f(x)| \leq \epsilon \text{ for all } k \geq K.$$

Uniform convergence on $[a, b]$ implies pointwise convergence on $[a, b]$.

Weierstrass Test

Let $f_k : \mathbb{R} \to \mathbb{R}$ be a sequence on functions defined on $[a, b]$. Suppose that there exists a sequence of numbers c_k such that

$$|f_k(x)| \leq c_k \text{ for all } x \in [a, b]$$

and $\sum_{k=1}^{\infty} c_k$ converges. Then $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function f on $[a, b]$.

Norm convergence

Let V be a vector space of functions f equipped with a norm $\|f\|$. We say a sequence of functions $f_1, \ldots, f_k$ converges to f in V if $f \in V$ and

$$\lim_{k\to\infty} \|f_k - f\| = 0.$$

In particular, consider the space of square-integrable functions on $[a, b]$ equipped with the L^2-norm. Then the mean square convergence is given by

$$\lim_{k\to\infty} \int_a^b \big[f_k(x) - f(x)\big]^2 dx = 0.$$

Parseval's Theorem

Let f be 2π-periodic, bounded and square-integrable. Then the Fourier series of f converges to f in the mean-squared sense and the following **Parseval's identity** holds:

$$\int_{-\pi}^{\pi} f^2(x)dx = \|f\|_2^2 = \frac{\pi}{2}a_0^2 + \pi \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2\right)$$

Vector Calculus

Vector Fields

A vector field is a function F that assigns each point in a subset of space a vector. In $\mathbb{R}^3$, we have

$$\begin{aligned} F(\mathbf{x}) &= F(x, y, z) \\ &= \big(F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\big) \\ &= F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k} \end{aligned}$$

We also define the vector differential operator ∇ in $\mathbb{R}^3$

$$\nabla := \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Divergence

If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, the divergence of F is the scalar-valued function

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Curl

If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, the curl of F is the vector-valued function

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Useful Identities

For functions f, g and vector fields $\mathbf{F}, \mathbf{G}$, we have the following identities

1. $\nabla(f + g) = \nabla f + \nabla g$
2. $\nabla(cf + g) = c\nabla f + \nabla g$, for a constant c
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ at x where $g(x) \neq 0$
5. $\text{div}(\mathbf{F} + \mathbf{G}) = \text{div } \mathbf{F} + \text{div } \mathbf{G}$
6. $\text{curl}(\mathbf{F} + \mathbf{G}) = \text{curl } \mathbf{F} + \text{curl } \mathbf{G}$
7. $\text{div}(f\mathbf{F}) = f\text{div } \mathbf{F} + \mathbf{F} \cdot \nabla f$
8. $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl } \mathbf{F} - \mathbf{F} \cdot \text{curl } \mathbf{G}$
9. $\text{div curl } \mathbf{F} = 0$ (curl of $\mathbf{F}$ is incompressible)
10. $\text{curl}(f\mathbf{F}) = f\text{curl } \mathbf{F} + \nabla f \times \mathbf{F}$
11. $\text{curl } \nabla f = 0$ (gradient field irrotational)
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g)$
13. $\text{div}(\nabla f \times \nabla g) = 0$
14. $\text{div}(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

Path Integrals

Let $c(t)$ be a parametrisation of a curve $\mathcal{C}$ for $a \le t \le b$. Assume that $f(x,y,z)$ and $c'(t)$ are continuous. Then

$$\int_{\mathcal{C}} f(x,y,z)ds = \int_a^b f\big(c(t)\big)\|c'(t)\|dt$$

The integral on the right is independent of parametrisation.

Properties of Path Integrals

- Additivity: $\displaystyle\int_{\mathcal{C}} (f_1 + f_2)ds = \int_{\mathcal{C}} f_1 \; ds + \int_{\mathcal{C}} f_2 \; ds$
- Scalar multiplication: $\displaystyle\int_{\mathcal{C}} \lambda f \; ds = \lambda \int_{\mathcal{C}} f \; ds$, $\lambda \in \mathbb{R}$

Line Integrals

Let $c(t)$ be a parametrisation of an oriented curve $\mathcal{C}$ that is continuously differentiable for $a \le t \le b$. The line integral of a vector field $\mathbf{F}$ along $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} \mathbf{F} \boldsymbol{\cdot} ds = \int_a^b \mathbf{F}(c(t)) \boldsymbol{\cdot} c'(t) \; dt$$

For $\mathbf{F} = (M, N, P) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$,

$$\int_{\mathcal{C}} \mathbf{F} \boldsymbol{\cdot} ds = \int_{\mathcal{C}} M dx + N dy + P dz$$

Properties of Line Integrals

Let $\mathcal{C}$ be a smooth oriented curve and let $\mathbf{F}$ and $\mathbf{G}$ be vector fields.

1. Linearity:

$$\int_{\mathcal{C}} (\mathbf{F} + \mathbf{G}) \boldsymbol{\cdot} ds = \int_{\mathcal{C}} \mathbf{F} \boldsymbol{\cdot} ds + \int_{\mathcal{C}} \mathbf{G} \boldsymbol{\cdot} ds$$

$$\int_{\mathcal{C}} \lambda \mathbf{F} \boldsymbol{\cdot} ds = \lambda \int_{\mathcal{C}} \mathbf{F} \boldsymbol{\cdot} ds, \; \lambda \in \mathbb{R}$$

2. Reversing orientation:

$$\int_{-C} \mathbf{F} \boldsymbol{\cdot} ds = -\int_{C} \mathbf{F} \boldsymbol{\cdot} ds$$

3. Additivity: If C is a union of n smooth curves $C_1, \ldots, C_n$, then

$$\int_{C} \mathbf{F} \boldsymbol{\cdot} ds = \int_{C_1} \mathbf{F} \boldsymbol{\cdot} ds + \ldots + \int_{C_n} \mathbf{F} \boldsymbol{\cdot} ds$$

Fundamental Theorem of Line Integrals

A vector field $\mathbf{F}$ is a gradient vector field if there exists a real valued function φ, called the potential function, such that $\mathbf{F} = \nabla\varphi$.

Fundamental Theorem for Gradient Vector Fields

If $\mathbf{F} = \nabla\varphi$ on a domain D, then for every oriented curve C in D with initial point P and terminal point Q,

$$\int_{\mathcal{C}} \mathbf{F} \boldsymbol{\cdot} ds = \varphi(Q) - \varphi(P)$$

Cross partials of a gradient vector field are equal

Let $\mathbf{F} = (F_1, F_2, F_3)$ be a gradient vector field whose components have continuous partial derivatives. Then the cross partials are equal:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

which is equivalent to curl $\mathbf{F} = \mathbf{0}$.

Green's Theorem

A *simple region* is both x-simple and y-simple.

Green's Theorem gives the relationship between the line integral around a simple closed curve C to a double integral over the region D bounded by C. There are two forms of Green's Theorem:

Flux-divergence or normal form

Let D be a bounded simple region in $\mathbb{R}^2$ with non-empty interior, whose boundary consists of a finite number of smooth curves.

Let C be the boundary of D with positive direction (anticlockwise). Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field that is continuously differentiable on D. Then the outward flux across is the double integral of the divergence, div $\mathbf{F}$

$$\begin{aligned}\oint_C (\mathbf{F} \boldsymbol{\cdot} \hat{\mathbf{n}}) \; ds &= \oint -N \; dx + M \; dy \\ &= \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial M}{\partial y}\right) dxdy\end{aligned}$$

Circulation-curl or tangential form

The counter-clockwise circulation of $\mathbf{F}$ over D is the double integral of the scalar curl, curl $\mathbf{F} \cdot \hat{k}$

$$\begin{aligned}\oint_C (\mathbf{F} \boldsymbol{\cdot} \hat{\mathbf{T}}) \; ds &= \oint_C M \; dx + N \; dy \\ &= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy\end{aligned}$$

Surface Integrals

Parametrised surfaces

A parametrised surface is a vector-valued function $\mathit{\Phi} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$\mathit{\Phi}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big).$$

The surface S corresponding to the function $\mathit{\Phi}$ is its image: $S = \mathit{\Phi}(D)$.

A parametrised surface is *smooth* if

(i) D is and elementary region

(ii) $\mathit{\Phi}$ is continously differentiable and one-to-one, except possibly on ∂D

(iii) S is regular, except possibly on ∂D

Examples

Some examples of parametrised surfaces that you will need to be familiar with include

1. A cone $z^2 = x^2 + y^2$ has parametrisation $\mathit{\Phi}(u,v) = (u\cos v, u \sin v, u)$, $0 \leq \theta \leq 2\pi$, $u \in \mathbb{R}$

2. A cylinder of radius R, $x^2 + y^2 = R^2$ has parametrisation $\mathit{\Phi}(\theta, z) = (R\cos\theta, R\sin\theta, z)$, $0 \leq \theta \leq 2\pi$, $z \in \mathbb{R}$

3. A sphere of radius R, $x^2 + y^2 + z^2 = R^2$ has parametrisation $\mathit{\Phi}(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\phi\sin\theta, R\cos\phi)$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$

Tangent vectors

$$T_u := \frac{\partial \mathit{\Phi}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$
$$T_v := \frac{\partial \mathit{\Phi}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

Then we have the normal n

$$\mathbf{n}(u,v) = \pm T_u \times T_v$$

The area of a surface S is

$$\text{Area}(S) = \iint_D \|n(u,v)\| \; dudv$$

The surface integral of f over S is given by

$$\iint_S f(x,y,z)\; ds = \iint_D f(\mathit{\Phi}(u,v))\|n(u,v)\| \; dudv$$

Integral Theorems

Stokes Theorem

Let S be a smooth oriented surface defined by a one-to-one parametrisation $\mathit{\Phi} : D \subseteq \mathbb{R}^2 \to S$. Let ∂S denote the oriented boundary of S and let $\mathbf{F}$ be a C^1 vector field on S. Then

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

(Gauss) Divergence Theorem

Let $W \subseteq \mathbb{R}^3$ be a bounded, solid and simple region, and let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$ which is continuously differentiable on W. Let S be the boundary of W which is a piecewise smooth parametrised surface formed by a finite union of oriented smooth surfaces (say S_i). Then,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \text{div } \mathbf{F} \; dV$$

where $\iint_S \mathbf{F} \cdot d\mathbf{S} = \sum \iint_{S_i} \mathbf{F} \cdot d\mathbf{S}$ and the surfaces are oriented such that the normal vector points outwards.

You will need to address the three key assumptions:

- The region $W \subseteq \mathbb{R}^3$ is *bounded, solid and simple*;
- The boundary S is *smooth* and *oriented such that the normal vector points outwards*;
- The vector field $\mathbf{F}$ is *continuously differentiable* on W.

We also have

$$\text{surface area of } S = \iiint_W \text{div } \hat{\mathbf{n}} \; dV.$$