MATH1231/1241 Revision Sheet

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We would like to preface this document by saying that this resource is first and foremost meant to be used as a reference and should NOT be used as a replacement for the course notes or lecture recordings. The course notes provided on Moodle are wonderfully written and contain an abundance of fully worked solutions and in depth explanations. Studying for this course using *only* this revision sheet would not be sufficient.

In addition, although the authors have tried their best to include everything essential taught in the course, it was ultimately up to their discretion on whether or not to include results/theorems/definitions etc. Anything that is missing is most definitely a conscious choice made by the authors.

Finally, any and all errors found within this document are most certainly our own. If you have found an error, please contact us via our Facebook page, or give us an email.

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Algebra

Vector Spaces

Note that, $\mathbb{F}$ is usually either $\mathbb{R}$ or $\mathbb{C}$. However, there are many other esoteric examples of what $\mathbb{F}$ could possibly be.

Definition of Vector Space

A vector space V over the field $\mathbb{F}$ is a set of vectors where addition and multiplication by a scalar are defined such that the following ten fundamental properties are satisfied:

1. **Closure under Addition.**

   If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.

2. **Associative Law of Addition.**

   If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \in V$, then $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

3. **Commutative Law of Addition.**

   If $\mathbf{u}$, $\mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

4. **Existence of Zero.**

   There exists an element $\mathbf{0} \in V$ such that, for all $\mathbf{v} \in V$, $\mathbf{v} + \mathbf{0} = \mathbf{v}$.

5. **Existence of Negative.**

   For each $\mathbf{v} \in V$ there exists an element $\mathbf{w} \in V$ (usually written as $-\mathbf{v}$), such that $\mathbf{v} + \mathbf{w} = 0$.

6. **Closure under Multiplication by a Scalar.**

   If $\mathbf{v} \in V$ and $\lambda \in \mathbb{F}$, then $\lambda v \in V$.

7. **Associative Law of Multiplication by a Scalar.**

   If $\lambda, \mu \in \mathbb{F}$ and $\mathbf{v} \in V$, then $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$.

8. If $\mathbf{v} \in V$ and $1 \in \mathbb{F}$ is the scalar one, then $1\mathbf{v} = \mathbf{v}$.

9. **Scalar Distributive Law.**

   If $\lambda, \mu \in \mathbb{F}$ and $\mathbf{v} \in V$, then $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$.

10. **Vector Distributive Law.**

    If $\lambda \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$.

For most intents and purposes, rarely do you have to deal with this (extensive) definition directly.

Roughly, a vector space is just something where we can multiply things by numbers and the regular laws of algebra that most people are inherently aware of, are satisfied.

Examples of Vector Spaces

1. $\mathbb{R}$ is a vector space over $\mathbb{R}$ or $\mathbb{Q}$.
2. $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ where $n \geq 2$.
3. $\mathbb{C}$ is a vector space over $\mathbb{C}$.
4. $\mathbb{C}^n$ is a vector space over $\mathbb{C}$ where $n \geq 2$.
5. The set of polynomials $\mathbb{P}$ is a vector space over any field $\mathbb{F}$.
6. The set of $m \times n$ matrices $\mathbb{M}_{mn}$ is a vector space over any field $\mathbb{F}$.
7. The set $\mathbb{F}^\infty$ which denote the space of infinite sequences that have finitely many elements which are nonzero is a vector space over any field $\mathbb{F}$ when scalar multiplication as well as addition is defined coordinate wise.

Subspaces

Proving that something is a vector space takes a lot of work, so we want to have a notion which allows us to recognise if something is a vector space without having to tediously go through all ten axioms.

We ask, "if we already have a known vector space, what subset can we take from this vector space so that we have yet another vector space?". This leads to the notion of a subspace:

- A subset S of a vector space V is a **subspace** if:
  - S is itself a vector space over the same field of scalars as V, and
  - S is under the same rules of addition and multiplication by scalars
- If there is at least one vector in V that is not in S, the subspace S is a **proper subspace** of V.

Subspace Theorem
A subset S of a vector space V over a field $\mathbb{F}$ is a subspace if and only if:

i) The vector $\mathbf{0}$ in V also belongs to S that is, $\mathbf{0} \in S$.

ii) S is closed under vector addition,

   If $\mathbf{a} \in S$, and $\mathbf{b} \in S$, it implies that $\mathbf{a} + \mathbf{b} \in S$.

iii) S is closed under multiplication by scalars,

   If $\mathbf{a} \in S$ and $\lambda \in \mathbb{F}$, then $\lambda\mathbf{a} \in S$.

Examples of Subspaces

1. The set of polynomials of degree 3 or less, $\mathbb{P}_3$, is a subspace of $\mathbb{P}$ over $\mathbb{F}$.

2. Any line or plane which passes through the origin in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ over $\mathbb{R}$.
3. The set $A \subseteq \mathbb{R}^3$ such that the coordinate entries of any element of A add up to 0 is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$.
4. The set D of $m \times n$ diagonal matrices is a subspace of $\mathbb{M}_{m\times n}$ over the field $\mathbb{R}$.

Linear Combinations

Let $S = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\} \subseteq V$. Then a **linear combination** of S is a sum of scalar multiples of the form

$$\lambda_1\mathbf{v_1} + \cdots + \lambda_n\mathbf{v_n} = \sum_{i=1}^{n} \lambda_i \mathbf{v_i}$$

with $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$.

- This definition really just says that a linear combination is just adding up a bunch of vectors with each vector having some scalar multiplied to it.
- It is crucial that S be a *finite* set.
- Any linear combination is always an element of V by the Subspace Theorem.

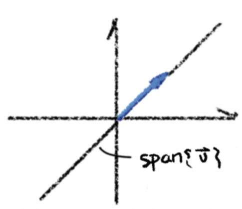
Spans

The span of the set $S \subseteq V$ is the set of all linear combinations of S. That is,

$$\begin{aligned}\operatorname{span}(S) &= \operatorname{span}\{\mathbf{v_1}, \ldots, \mathbf{v_n}\} \\ &= \left\{ \mathbf{v} \in V : \mathbf{v} = \sum_{i=1}^{n} \lambda_i \mathbf{v_i} \text{ for } \lambda_i \in \mathbb{F} \right\}.\end{aligned}$$

- The span of a set S is always a subspace of V. Further, $\operatorname{span}(S)$ is the smallest subspace containing S.
- We say that $S \subseteq V$ is called a spanning set for V if $\operatorname{span}(S) = V$ or, equivalently, if every vector in V can be expressed as a linear combination of vectors in S.

The subspace of $\mathbb{R}^m$ spanned by the columns of an $m \times n$ matrix A is called the column space of A, and is denoted by $\operatorname{col}(A)$. That is, $\operatorname{col}(A)$ is the set of all linear combinations of the columns in the matrix A.



Linear independence

Suppose that $S = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\} \subseteq V$. The set S is **linearly independent** if the only solution to

$$\lambda_1\mathbf{v_1} + \cdots + \lambda_n\mathbf{v_n} = 0$$

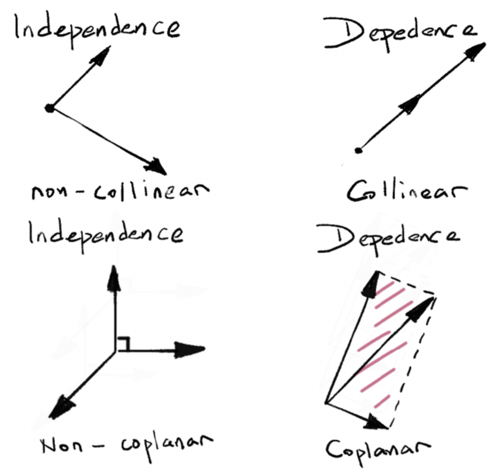
is,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

A set is considered **linearly dependent** if there exists another solution, where the scalars are *not all* zero.

Properties of Linear Independence

- A linear combination $v_i \in S$ is *unique* if and only if S is a linearly independent set.
- A set S is linearly independent if and only if no vector in S is in the span of other vectors in S.
- The span of every proper subset of S is a proper subspace of $\operatorname{span}(S)$ if and only if S is a linearly independent set.
- Suppose that $S \subseteq V$ is a finite linearly independent set, and $\mathbf{v} \in V$. If $\mathbf{v} \notin \operatorname{span}(S)$ then $S \cap \{\mathbf{v}\}$ is a linearly independent set.
- Suppose that S is a spanning set of V and V' is any linearly independent set. If $|V| = n$ and $|V'| = m$, it is always the case that $n \geq m$.



Bases and dimension

A set of vectors $B \subseteq V$ is called a basis for V if:

1. B is a linearly independent set, and

2. $\text{span}(B) = V$.

If $|B| = n$ for a vector space V, then we say that the *dimension* of V is n. Denoted as $\dim(V) = n$.

Properties of Bases

- Suppose that $B_1 = \{\mathbf{u_1}, \ldots, \mathbf{u_n}\}$ and $B_2 = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ are two bases for the same vector space V, then $m = n$.

- Suppose that V is a finite dimensional vector space.

  1. the number of vectors in any spanning set for V is greater than or equal to the dimension of V;
  2. the number of vectors in any linearly independent set in V is less than or equal to the dimension of V;
  3. if the number of vectors in a spanning set is equal to the dimension then the set is also a linearly independent set and hence a basis for V;
  4. if the number of vectors in a linearly independent set is equal to the dimension then the set is also a spanning set and hence a basis for V.

- If S is non-empty and finite, then S contains a subset which is a basis for $\text{span}(S)$.

- If V is any non-zero vector space which can be spanned by a finite set of vectors, then V has a basis.

- Every linearly independent subset of a vector space V, can be extended to a basis for V.

How to reduce a spanning set S of $\mathbb{R}^m$ to a basis in $\mathbb{R}^m$

1. Create a matrix A whose columns are members of S.

2. Row reduce A until you get a matrix A', where A' is the row-echelon form of A.

3. Create a new set S' which consists only of the leading columns in A'.

4. Then, S' is a basis for $\text{span}(S)$.

How to extend a linearly independent set to a basis in $\mathbb{R}^m$

Suppose that the linearly independent set is $S = \{v_1, \ldots, v_n\}$.

1. Create a matrix A whose columns are members of S, then followed by the members of the standard basis for $\mathbb{R}^m$.

2. Row reduce A until you get a matrix A', where A' is the row-echelon form of A.

3. Create a new set S' which consists only of the leading columns in A'.

4. S' is a basis for $\mathbb{R}^m$ containing S as a subset.

Coordinate Vectors

Let V be an n-dimensional vector space and let the ordered set of vectors $B = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ be a basis for V. If

$$\mathbf{v} = x_1\mathbf{v_1} + \cdots + x_n\mathbf{v_n}$$

then the vector

$$[\mathbf{v}]_B = x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is called the **coordinate vector of v with respect to the ordered basis *B*.**

Properties of Coordinate Vectors

If B is an ordered basis for a vector space V over a field $\mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in \mathbb{F}$, then

- $\mathbf{u} = \mathbf{v}$ if and only if $[\mathbf{u}]_B = [\mathbf{v}]_B$, that is, two vectors are equal if and only if the corresponding coordinate vectors are equal.

- $[\mathbf{u} + \mathbf{v}]_B = [\mathbf{u}]_B + [\mathbf{v}]_B$, that is, the coordinate vector of the sum of two vectors is equal to the sum of the two corresponding coordinate vectors.

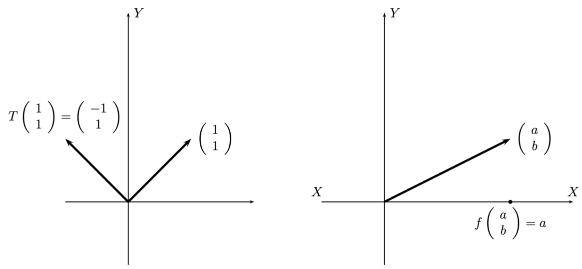
- $[\lambda\mathbf{u}]_B = \lambda[\mathbf{u}]_B$, that is, the coordinate vector of a scalar multiple of a vector is equal to the same scalar multiple of the corresponding coordinate vector.

Linear Transformations

Linear Maps

Let V and W be two vector spaces over the same field $\mathbb{F}$. A function $T : V \to W$ is called a **linear map** or **linear transformation** if the following two conditions are satisfied.

- **Addition Condition.** $T(\mathbf{v}+\mathbf{v}') = T(\mathbf{v})+T(\mathbf{v}')$ for all $\mathbf{v}, \mathbf{v}' \in V$, and
- **Scalar Multiplication Condition.** $T(\lambda\mathbf{v}) = \lambda T(\mathbf{v})$ for all $\lambda \in \mathbb{F}$ and $\mathbf{v} \in V$.



Properties of Linear Maps

If T is a linear map with domain V and S is a set of vectors in V , then the function value of a linear combination of S is equal to the linear combination of the function values of S, that is,

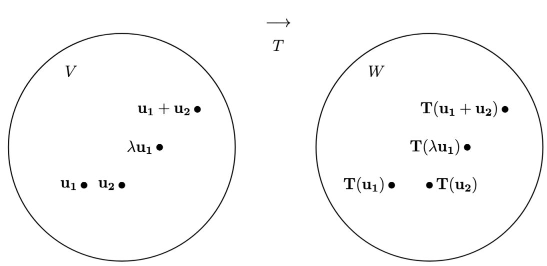
$$T(\lambda_1\mathbf{v}_1 + \cdots + \lambda_n\mathbf{v}_n) = \lambda_1 T(\mathbf{v}_1) + \cdots + \lambda_n T(\mathbf{v}_n).$$

where $S = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ and $\lambda_1, ..., \lambda_n$ are scalars.

Further, if $B = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a basis for the domain V then for all $\mathbf{v} \in V$ we have

$$T(\mathbf{v}) = x_1 T(\mathbf{v}_1) + \cdots + x_n T(\mathbf{v}_n),$$

where $x_1, ..., x_n$ are the scalars in the unique linear combination $\mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$ of the basis B.

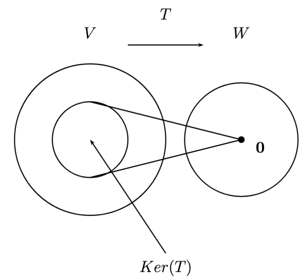


Subspaces associated with linear maps

The kernel of a map
Let $T : V \to W$ be a linear map. Then the **kernel** of T (written $\ker(T)$) is the set of all values in V that is sent to 0 when passed through T , i.e.

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = 0\}.$$

The **nullity** of a linear map T is the dimension of $\ker(T)$.



Show that a vector $\boldsymbol{v}$ *is in the kernel of a linear map* T *simply by verifying* $T(\boldsymbol{v}) = 0$.
In particular, $\boldsymbol{0} \in ker(T)$ *for any linear map* T *, since* $T(\boldsymbol{0}) = \boldsymbol{0}$.

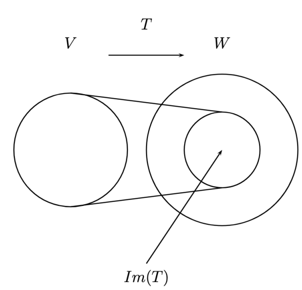
Theorem. If $T : V \to W$ is a linear map, then $\ker(T)$ is a subspace of the domain V.

For a matrix A:
$\text{nullity}(A)$ = maximum number of independent vectors in the solution space of $A\mathbf{x} = 0$
= number of parameters in the solution of $A\mathbf{x} = 0$ obtained by Gaussian elimination
= number of non-leading columns in an equivalent row-echelon form U for A.

Image
Let $T : V \to W$ be a linear map. Then the image of T is the set of all function values of T , that is, it is the subset of the codomain W defined by

$$\text{im}(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}.$$

The **rank** of a linear map T is the dimension of $\text{im}(T)$.



For a matrix A:
$\text{rank}(A)$ = maximal number of linearly independent

columns of A
$=$ number of leading columns in a row-echelon form U for A.

Rank, nullity and solutions of $\mathbf{Ax} = \mathbf{b}$

(Rank-Nullity Theorem for Matrices). For any matrix A,

$$\text{rank}(A) + \text{nullity}(A) = \text{number of columns of } A.$$

(Rank-Nullity Theorem). Suppose V and W are finite dimensional vector spaces and

$$\text{rank}(A) + \text{nullity}(A) = \dim(V).$$

Eigenvalues and Eigenvectors

Definition of eigenvectors and eigenvalues

Let $T : V \to V$ be a linear map. Then if a scalar λ and non-zero vector $\mathbf{v} \in V$ satisfy

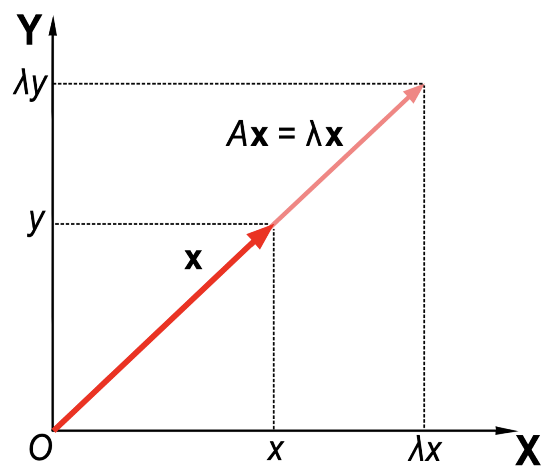
$$T(\mathbf{v}) = \lambda\mathbf{v}$$

then λ is an eigenvalue of T and $\mathbf{v}$ is an eigenvector of T for the eigenvalue λ.

Similarly, let $A \in \mathbb{C}^{n\times n}$ be a square matrix. Then if a scalar $\lambda \in \mathbb{C}$ and non-zero vector $\mathbf{v} \in \mathbb{C}^n$ satisfy

$$A\mathbf{v} = \lambda\mathbf{v}$$

then λ is an eigenvalue of A and $\mathbf{v}$ is an eigenvalue of A for the eigenvalue λ.



Finding eigenvectors and eigenvalues

- λ is an eigenvalue of a square matrix A if and only if $\det(A - \lambda I) = 0$.

- $\mathbf{v}$ is an eigenvector of A with eigenvalue λ if and only if $\mathbf{v} \neq 0$ and $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

Eigenspace

For a square matrix A, the eigenspace of the eigenvalue λ is the set

$$E_\lambda = \ker(A - \lambda I)$$

i.e. the set of all eigenvectors with eigenvalue λ together with $\mathbf{0}$.

Characteristic polynomial

If A is an $n \times n$ matrix and $\lambda \in \mathbb{C}$, then

$$p(\lambda) = \det(A - \lambda I)$$

is a complex polynomial of degree n in λ, called the characteristic polynomial for the matrix A.

- An $n \times n$ matrix A has exactly n eigenvalues in $\mathbb{C}$ (counting multiplicities). These eigenvalues are the zeroes of the characteristic polynomial.

Diagonalisation

If an $n \times n$ matrix has n distinct eigenvalues, then it has n linearly independent eigenvectors, which form a basis for $\mathbb{C}^n$.

- The converse is not true in general.

A square matrix A is said to be a diagonalisable matrix if there exists an invertible matrix M and diagonal matrix D such that

$$M^{-1}AM = D.$$

An $n \times n$ matrix A is diagonalisable if and only if it has n linearly independent eigenvectors.

- The diagonal elements of D are the eigenvalues of A.
- The jth column of M is the eigenvector of A corresponding to the jth element of the diagonal of D.

Eigenvalues of symmetric matrices

Suppose that A is an $n \times n$ symmetric real matrix. Then

- The eigenvalues of A are real.
- The eigenvectors can be chosen to form an orthonormal basis for $\mathbb{R}^n$.
- Note that having distinct eigenvalues is a sufficient but not necessary condition for a matrix to be diagonalisable.

Applications

- Let A be an $n \times n$ diagonalisable matrix and $k \in \mathbb{N}$. Then

$$A^k = MD^kM^{-1}.$$

- $\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$ is a solution of the differential equation

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = A\mathbf{y}$$

if and only if $\mathbf{v}$ is an eigenvector of A for the eigenvalue λ.
If A is diagonalisable, then the general solution to the differential equation is of the form

$$\mathbf{y}(t) = \sum_{k=1}^{n} \alpha_k e^{\lambda_k t}\mathbf{v}_k$$

where $\mathbf{v}_k$ are linearly independent eigenvectors with corresponding eigenvalues $\lambda_k, k = 1, \dots, n$.

- Diagonalization can be used to efficiently compute the powers of a matrix $A = PDP^{-1}$ since it only involves the powers of a diagonal matrix. For example, for the matrix A with eigenvalues $\lambda = 1,\ 1,\ 2$ we compute:

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & 2^k \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2-2^k & -1+2^k & 2-2^{k+1} \\ 0 & 1 & 0 \\ -1+2^k & 1-2^k & -1+2^{k+1} \end{bmatrix}.$$

where

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & 3 \end{bmatrix}.$$

Probability and Statistics

Probability Definitions

- The set of all possible outcomes of a given experiment is called the sample space.
- An event A is a subset of a sample space.

Suppose S is a sample space, then the probability is a function P defined on the set of all events such that

1. $0 \leq P(A) \leq 1$ for all $A \subseteq S$
2. $P(\emptyset) = 0$
3. $P(S) = 1$
4. if A, B are mutually exclusive events then

$$P(A \cup B) = P(A) + P(B).$$

Properties

Suppose A, B are events.

- $P(A^c) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If $A \subseteq B$, then $P(A) \leq P(B)$.

Conditional Probability

Let A and B be events such that $P(B) \neq 0$. Then the probability of A given B is

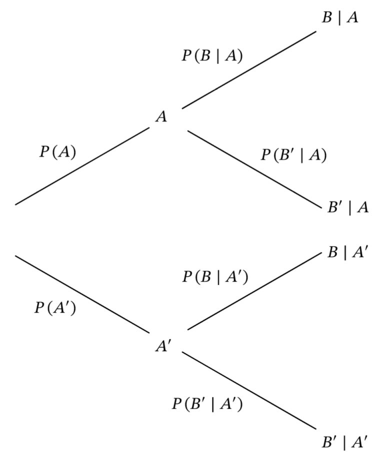
$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

If event A always occurs with exactly one of $B_1, \ldots, B_k$, then we have,
Total probability rule

$$P(A) = \sum_{i=1}^{k} P(A|B_i)P(B_i)$$

and **Bayes rule**

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}.$$



Independence

Events A and B are mutually independent if

$$P(A \cap B) = P(A)P(B).$$

- A and B are mutually independent if and only if $P(A|B) = P(A)$.

Random Variables

A random variable on a sample space S is a real-valued function $X : S \to \mathbb{R}$.
A discrete random variable is one whose values are countable. A **probability distribution** for a discrete random variable X with values $\{x_k : k \in \mathbb{Z}\}$ is a set of numbers $\{p_k : k \in \mathbb{Z}\}$ such that

$$P(X = x_k) = p_k \geq 0 \quad \text{and} \quad \sum_{k=-\infty}^{\infty} p_k = 1.$$

The **cumulative distribution function** for a discrete random variable is the function $F_X(x) : \mathbb{R} \to \mathbb{R}$

$$F_X(x) = P(X \leq x) = \sum_{k : x_k \leq x} p_k, \quad x \in \mathbb{R}.$$

- F_X is a non-decreasing function.
- If $a \leq b$ then $P(a < X \leq b) = F_X(b) - F_X(a)$.
- $$\lim_{x \to -\infty} F_X(x) = 0 \quad \text{and} \quad \lim_{x \to -\infty} F_X(x) = 1.$$

Expected value

The mean or expected value of a discrete random variable X is

$$\mathbb{E}(X) = \sum x_k p_k.$$

- Expected value is linear, i.e. suppose a, b are constants and X, Y random variables, then

  $$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y).$$

  Note this is true even if X and Y are not independent.

- If $Y = g(X)$, then

  $$\mathbb{E}(Y) = \sum g(x_k) p_k.$$

Variance and Standard Deviation

The variance of a random variable X is

$$\mathrm{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

The standard deviation of X is

$$\sigma_X = \sqrt{\mathrm{Var}(X)}.$$

Part II
Calculus

Functions of Several Variables

Sketching simple surfaces in $\mathbb{R}^3$

Definition: A contour or level curve of a function $F : \mathbb{R}^2 \to \mathbb{R}$ is a curve in $\mathbb{R}^2$ corresponding to an equation of the form $F(x, y) = C$, where C is a constant.

Partial Differentiation

Definition: Suppose that F is a function of two variables x and y. The partial derivatives of F with respect to x and y are defined by

$$F_x(x, y) = \lim_{h \to \infty} \frac{F(x + h, y) - F(x, y)}{h}$$

and

$$F_y(x, y) = \lim_{h \to \infty} \frac{F(x, y + h) - F(x, y)}{h}$$

wherever these limits exist.

The mixed derivative theorem: Suppose that F is a function of two variables. If F and all its first and second order partial derivatives are continuous then

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

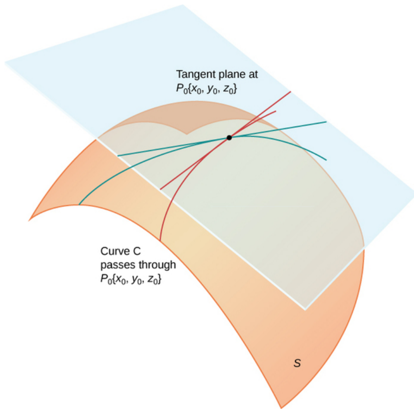
Tangent planes to surfaces

Theorem: Suppose that F is a function of two variables and (x_0, y_0, z_0) is a point that lies on the surface $z = F(x, y)$. If the surface has a tangent plane at the point (x_0, y_0, z_0), then the tangent plane is given by the equation

$$z = z_0 + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$$

and a normal vector to the surface at (x_0, y_0, z_0) is given by

$$\begin{pmatrix} F_x(x_0, y_0) \\ F_y(x_0, y_0) \\ -1 \end{pmatrix}.$$



The total differential approximation

Formula for total differential approximation.

$$\Delta F \approx \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y.$$

Chain rules

Theorem: Suppose that F is a function of two variables and that x and y are both functions of one variable. Define the function ϕ by $\phi(t) = F(x(t), y(t))$ and the point (x_0, y_0) by $(x_0, y_0) = (x(t_0), y(t_0))$. If x and y are both differentiable at t_0 and the partial derivatives of F exist and are continuous at (x_0, y_0), then ϕ is differentiable at t_0 and

$$\phi'(t_0) = D_1F(x_0, y_0)x'(t_0) + D_2F(x_0, y_0)y'(t_0).$$

Functions of more than two variables

The partial derivatives of F are defined by

$$F_x(x, y, z) = \lim_{h\to\infty} \frac{F(x+h, y, z) - F(x, y, z)}{h}$$

$$F_y(x, y, z) = \lim_{h\to\infty} \frac{F(x, y+h, z) - F(x, y, z)}{h}$$

$$F_z(x, y, z) = \lim_{h\to\infty} \frac{F(x, y, z+h) - F(x, y, z)}{h}$$

The total differentiation approximation ΔF is given by

$$\Delta F \approx \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y + \frac{\partial F}{\partial z}\Delta z.$$

Integration Techniques

Trigonometric integrals

The first class of trigonometric integrals considered consist of integrals of the form

$$\int \cos^m x \sin^n x dx,$$

where m and n are non-negative integers. There are essentially two cases: (i) either m or n (or both) are odd; or (ii) both m and n are even.

Case (i). Suppose that m is odd. Then we use the substitution u = sin x along with the identity

$$\sin^2 x + \cos^2 x = 1$$

to evaluate the integral.

Case (ii). The case where both m and n are even requires an entirely different approach. This time we use the identities

$$\cos^2 x = \frac{1+\cos 2x}{2} \text{ and } \sin^2 x = \frac{1-\cos 2x}{2}$$

to change integral into a sum of integrals of the form

$$\int \cos^k 2x dx.$$

We then repeat the methods of Case (i) or Case (ii) until each integral in the sum is easy to compute.

Sums to products

The next class of trigonometric integrals consists of integrals of the form

$$\int \cos mx \sin nx dx, \int \cos mx \cos nx dx \text{ or}$$

$$\int \sin mx \sin nx dx$$

To evaluate the integrals we need the following trigonometric identities.

Theorem: Suppose that A and B are real numbers. Then

$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$

$$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B)).$$

Reduction Formulae

A reduction formula is a formula connecting terms within a sequence of integrals $\{I_n\}_{n=1}^{\infty}$. For instance, for the integral

$$I_n = \int x^n e^x dx,$$

we may have the recurrence relation

$$I_n = x^n e^x - nI_{n-1}.$$

The primary technique in these question is to use integration by parts. We begin with one of the integrals (often I_n) and work out a choice of u and dv that will reduce I_n to an integral involving I_{n-1}.

For instance, in this example, we would want to set $u = x^n$ and $dv = x^n dx$ so that in the term $\int u dv = uv - \int v du$, the power of x decreases to $n-1$. This example would yield

$$\begin{aligned}\int x^n e^x dx &= x^n e^x - \int nx^{n-1} e^x dx \\ &= x^n e^x - n \int x^{n-1} e^x dx \\ &= x^n e^x - nI_{n-1}.\end{aligned}$$

Trigonometric and hyperbolic substitutions

Many integrals can be evaluated by finding the right substitution, but unfortunately there is no general systematic way to do this. Integrals involving square roots of quadratics often yield to trigonometric or hyperbolic substitutions.

The following table indicates which substitution can be tried for integrals containing an expression of the form $\sqrt{\pm x^2 \pm a^2}$.

Expression in integral	Trigonometric substitution	Hyperbolic substitution
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$	$x = a \tanh\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$	$x = a \sinh\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$	$x = a \cosh\theta$

Whether or not a trigonometric substitution is more efficient than a hyperbolic substitution depends on the particular integral. In general, trigonometric substitutions are favoured because once integration is completed in the variable θ, it is easier to restate the result in terms of x.

Integrating rational functions

In this subsection we give an overview of the approach to integrating rational functions. The basic procedure is summarised below.

1. If the rational function is improper, then use polynomial division to write f as the sum of a polynomial and a proper rational function. Since the polynomial is easy to integrate, we need only focus on integrating a proper rational function.

2. It can be shown using algebra that every proper rational function f can be written as a unique sum of functions of the form

$$\frac{A}{(x-a)^k} \text{ and } \frac{Bx + C}{(x^2 + bx + c)^k}$$

   where the quadratic $x^2 + bx + c$ is irreducible. This sum is called the partial fractions decomposition of f.

3. Now we only need to integrate functions of the form as shown above. By completing the square, using a substitution or performing simple algebraic manipulation, these can be integrated by the standard formulae

$$\int x^k dx = \frac{x^{k+1}}{k+1} + C, k \neq -1$$

$$\int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\frac{x}{a} + C.$$

Ordinary Differential Equations

An introduction

Definition: An ordinary differential equation is expressed in terms of exactly one independent variable and one (or more) of the derivatives of a function of this variable. The order of an ordinary differential equation is the order of the highest derivative present.

Definition: A solution to an nth order ordinary differential equation is a function which is n-times differentiable and satisfies the given equation.

Initial value problems

Definition: An initial value problem is an nth order ODE together with a set of values of the solution and its first $(n-1)$ derivatives at some fixed point x_0. These values are called the initial conditions of the initial value problem. For instance, an example of a 4th degree initial value problem is

$$y'''' = 3x^2 + 4xy,$$

with

$$y(1) = 5, y'(1) = 3, y''(1) = -2, y'''(1) = 4.$$

Separable ODEs

A separable ODE is a differential equation where the two variables involved (x and y) can be separated so that all the ys are on one side of the equation and all xs are on the other.

First order linear ODEs

A first order linear ODE can be written in the standard form

$$\frac{dy}{dx} + f(x)y = g(x).$$

where f and g are given functions of a single variable x. The ODE is called linear since there are no non-linear terms (such as y^2, $\sin y$ or $\sqrt{y'}$) involving y or its derivative y'. A method for solving first order linear ODEs is summarised in the steps below.

1. Write the ODE in the standard form.
2. Calculate $e^{\int f(x)dx}$ (ignoring the constant of integration). We denote this by $h(x)$ and call it the integrating factor.
3. Multiply the standard form by the integrating factor $h(x)$ to obtain

   $$h(x)\frac{dy}{dx} + h(x)f(x)y = g(x)h(x).$$

   By using the product rule for differentiation, the left-hand side can now be rewritten so that

   $$\frac{d}{dx}(h(x)y) = g(x)h(x).$$

4. Integrate both sides and then rearrange for y to solve the ODE. Don't forget the constant of integration!

Exact ODEs

Definition: An ordinary differential equation of the form

$$F(x,y) + G(x,y)\frac{dy}{dx} = 0$$

is called exact if

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}.$$

Solving ODEs by using a change of variable (MATH1241 ONLY)

We can use substitutions such as $y(x) = xv(x)$ and $z(x) = \frac{1}{y(x)}$ to solve ODEs. The aim in these kinds of questions is to express everything in terms of the new, introduced function of x, for example $v(x)$ in the first case.

Special case: $y(x) = xv(x)$

We use this case when we have an ODE in the form

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right).$$

In this case, we have that

$$\frac{dy}{dx} = x\frac{dv}{dx} + v,$$

so that

$$x\frac{dv}{dx} + v = g(v),$$

that is,

$$\frac{dv}{dx} = \frac{g(v) - v}{x},$$

which is a separable equation that can be solved using standard techniques.

Second order linear ODEs with constant coefficients

Definition: A second order linear ODE with constant coefficients is said to be homogeneous if it is of the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0,$$

where a and b are real numbers.

Definition: The characteristic equation of the second order linear ODE

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0,$$

is given by

$$\lambda^2 + a\lambda + b = 0.$$

Solving second order homogeneous ODEs

Case (i): The characteristic equation has two distinct real roots λ_1 and λ_2. Hence the general solution is given by

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

where A and B are real numbers.

Case (ii): The characteristic equation has a repeat real real root λ_1. Hence the general solution is given by

$$y = Ae^{\lambda_1 x} + Bxe^{\lambda_1 x}$$

where A and B are real numbers.

Case (iii): The characteristic equation has two distinct complex roots $\alpha + \beta i$ and $\alpha - \beta i$, where α and β are real numbers and $\beta \neq 0$. Hence the general solution in this case is given by

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

where A and B are real numbers.

Solving second order non-homogeneous ODEs

1. Find the solution y_H to the corresponding homogeneous equation (by first identifying the roots of the characteristic equation).

2. Find a particular solution y_P to the second order ODE by guessing the form of the solution and solving for the undetermined coefficients.

3. The general solution y is then given by $y = y_H + y_P$.

Given a function f, the following table indicates which guess for y_P will always yield a particular solution for the non-homogeneous ODE.

$f(x)$	Guess for y_P
$P(x)$ (nth-degree polynomial)	$Q(x)$ (nth-degree polynomial)
$P(x)e^{sx}$	$Q(x)e^{sx}$
$P(x)\cos(sx)$	$Q_1(x)\cos(sx) +$ $Q_2(x)\sin(sx)$
$P(x)\sin(sx)$	$Q_1(x)\cos(sx) +$ $Q_2(x)\sin(sx)$
$P(x)e^{sx}\cos(tx)$ or $P(x)e^{sx}\sin(tx)$	$Q_1(x)e^{sx}\cos(tx) +$ $Q_2(x)e^{sx}\sin(tx)$

Essentially this table indicates that a particular solution y_P will have the form indicated on the right hand side of the table, with the coefficients of the polynomial being undetermined. Where the degree of the polynomial $P(x)$ on the left hand side is n, the degree of the polynomials $Q_i(x)$ on the right hand side are n.

Note that if any term of the guess for y_P is a solution to the homogeneous ODE, then we must multiply the guess for y_P by x to obtain the general solution. If any term of the new guess is still a solution to the homogeneous ODE, then we multiply by x again.

Taylor Series

Taylor polynomials

Suppose that f is n-times differentiable at a. Then the Taylor polynomial p_n of degree n for f about a is given by $p_n(x) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \cdots + \frac{f^{(n)}(a)}{n!}x^n$. We also call p_n the nth Taylor polynomial for f about a.

Taylor's Theorem

Suppose that f has $n+1$ continuous derivatives on an open interval I containing a. Then for each x in I,

$$f(x) = p_n(x) + R_{n+1}(x),$$

where p_n is the nth Taylor polynomial about a and the remainder $R_{n+1}(x)$ is given by

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt.$$

Lagrange formula for the remainder

Suppose that f has $n+1$ continuous derivatives on an open interval I containing a. Then for each x in I,

$$f(x) = p_n(x) + R_{n+1}(x),$$

where p_n is the nth Taylor polynomial about a and the remainder $R_{n+1}(x)$ is given by

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some real number c between a and x.

Classifying stationary points

Suppose that f is n times differentiable at a and that $f(a) = 0$. If

$$f''(a) = f'''(a) = ... = f^{(k1)}(a) = 0,$$

but $f^{(k)}(a) \neq 0$, where $k \leq n$, then

1. a is a local minimum point if k is even and $f^{(k)}(a) > 0$;

2. a is a local maximum point if k is even and $f^{(k)}(a) < 0$;

3. a is an horizontal point of inflexion if k is odd.

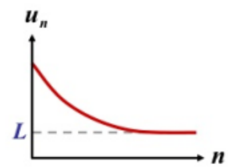
Sequences

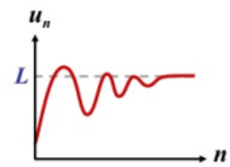
A sequence is a real-valued function defined on a subset of the natural numbers. Sequences are usually denoted by $\{a_n\}$ where the number a_n is called the nth term of the sequence.

Describing the limiting behaviour of sequences

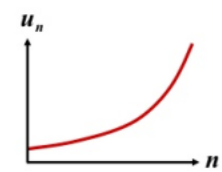
Suppose that $\{a_n\}$ is a sequence.

- If a_n approaches some finite number L, we say that the sequence $\{a_n\}$ is *convergent* and write $lim_{a_n \to \infty} a_n = L$





- If the sequence $\{a_n\}$ is not convergent, we say that $\{a_n\}$ is *divergent.*



Divergent sequences can be further classified according to the list below.

- If $a_n \to \infty$ as $n \to \infty$ (that is, an grows without bound) then we say that the sequence diverges to *infinity.*

- If $a_n \to -\infty$ as $n \to \infty$ then we say that the sequence diverges to *negative infinity.*

- If $\{a_n\}$ has no limit as $n \to \infty$ but remains bounded then we say that $\{a_n\}$ is *boundedly divergent.*

- If $\{a_n\}$ exhibits none of the above behaviour then we say that $\{a_n\}$ is *unboundedly divergent.*

Techniques for calculating limits of sequences

*Many of the rules and techniques given in MATH1131 for calculating limits of functions including **standard arithmetic operations** and the **pinching theorem** also apply for limits of sequences.*

The following table compares the growth of various sequences as $n \to \infty$.

a_n	growth rate as $n \to \infty$
1	constant: does not grow
$\ln n$	grows slowly
n^k, where $k > 0$	growth rate is faster for larger k
c^n, where $c > 1$	growth rate is faster for larger c
$n!$	grows rapidly
n^n	grows very rapidly

If $\{a_n\}_{n=0}^{\infty}$ is a bounded monotonic sequence of real numbers then it converges to some real number L.

Infinite series

Suppose that $\{a_k\}_{k=0}^{\infty}$ is a sequence of real numbers. For each natural number n, let s_n denote the nth partial sum given by

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{k=0}^{n} a_k.$$

If the sequence $\{s_n\}_{k=0}^{\infty}$ of partial sums converges to a number L then we say that the infinite series $\sum_{k=0}^{n} a_k$ *converges* to L and we write

$$\sum_{k=0}^{\infty} a_k = L.$$

In this case we also say that the series is *summable*. If the sequence $\{s_n\}_{k=0}^{\infty}$ of partial sums diverges then we say that the infinite series $\sum_{k=0}^{n} a_k$ diverges.

Tests for series convergence

The kth term divergence test

If $a_k \not\to 0$ as $k \to \infty$ then $\sum_{k=0}^{\infty} a_k$ diverges.

The kth term divergence test is equivalent to the following theorem.

If $\sum_{k=10}^{\infty} a_k$ converges then $a_k \to 0$ as $k \to \infty$.

Note that this theorem cannot be used to show that a series converges; it is more akin to a diagnostic test to show when a series does not converge.

The integral test

Suppose that $\sum_{k=1}^{\infty} a_k$ is an infinite series with positive terms.
Suppose $f(x)$ is a positive integrable function decreasing on $[1, \infty)$ such that for each positive integer where k, $f(k) = a_k$.

1. If $\int_1^{\infty} f(x)\,dx$ converges then so does $\sum_{k=1}^{\infty} a_k$.
2. If $\int_1^{\infty} f(x)\,dx$ diverges then so does $\sum_{k=1}^{\infty} a_k$.

Convergence and divergence of p-series

The series
$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$
converges if $p > 1$ and diverges if $p \leq 1$. This group of series can be used as an appropriate benchmark, combined with other tests, to show that more complicated series converge.

The comparison test

Suppose that $\{a_k\}_{k=1}^{\infty}$ and $\{b_1\}_{k=0}^{\infty}$ are two positive sequences such that $a_k \leq b_k$ for every $k = 1, 2, \ldots$.

1. If $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ also converges.
2. If $\sum_{k=1}^{\infty} a_k$ diverges then $\sum_{k=1}^{\infty} b_k$ also diverges.

The limit form of the comparison test

Suppose a_n, b_n are sequences with positive terms and suppose $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite and **not** zero. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

The ratio test

Suppose that $\sum a_n$ is an infinite series with positive terms and that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r.$$

1. If $r < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $r > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

No conclusion can be drawn from the case $r = 1$. In this case the ratio test would not be appropriate.

Leibniz' test for alternating series

Suppose that $\{a_k\}_{k=0}^{\infty}$ is a sequence of real numbers satisfying the following properties:

1. $a_k \geq 0$;
2. $a_k \geq a_{k+1}$ for all k (that is, the sequence is non-increasing); and
3. $\lim_{k\to\infty} a_k = 0$

Then the alternating series $\sum_{k=0}^{\infty} (-1)^k a_k$ converges.

Corollary: If the value of the convergent series $\sum_{k=0}^{\infty} (-1)^k a_k$. is L and the nth partial sum of the same series is s_n, then

$$|s_n L| \leq a_{n+1}$$

for every natural number n.

Absolute and conditional convergence

A series $\sum_{k=0}^{\infty} a_k$ is said to be absolutely convergent if the series $\sum_{k=0}^{\infty} |a_k|$ is convergent.

A series is conditionally convergent if it converges but does not converge absolutely.

Taylor series

Suppose that a function f has derivatives of all orders at a. Then the series

$$f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f^{(3)}(a)}{3!}x^3 + \dots,$$

which may also be written as

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the *Taylor series* for f about a. In the case when $a = 0$, the series is also called the *Maclaurin series* for f.

The following formulae hold whenever x lies in the given interval.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots x \in (-1, 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad x \in \mathbb{R}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad x \in \mathbb{R}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad x \in \mathbb{R}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \quad x \in \mathbb{R}$$

$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \dots \qquad x \in \mathbb{R}$$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \dots \quad x \in (-1, 1]$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad x \in [-1, 1]$$

Moreover, if x lies outside the given interval, then the corresponding Maclaurin series diverges.

Power series

Suppose that $\{a_k\}_{k=0}^{\infty}$ is a sequence of real numbers and that $a \in \mathbb{R}$. A series of the form

$$\sum_{k=0}^{\infty} a_k x^k$$

is called a *power series in powers of x*. A series of the form

$$\sum_{k=0}^{\infty} a_k (x-a)^k$$

is called a *power series in powers of $x - a$*.

Radius of convergence

If a power series in powers of $x - a$ converges at all points in some interval $(a - R, a + R)$, then the number R is called the *radius of convergence* for the power series.

The interval $(a - R, a + R)$ is called the *open interval of convergence* for the power series.

If the power series converges for all real x, we say that the radius of convergence is infinite.

Suppose that $\{a_k\}_{k=0}^{\infty}$ is a sequence of real numbers such that $\lim_{k\to\infty} |\frac{a_k}{a_{k+1}}| = R$ for some real number R. Then the power series in powers of $x - a$

- *converges absolutely whenever $|x - a| < R$, and*
- *diverges whenever $|x - a| > R$.*

One can deduce the convergence at each endpoint by substituting the endpoint into the power series.

Manipulation of power series

Suppose that a power series $\sum_{k=0}^{\infty} a_k x^k$ converges in the interval $(-R, R)$, where R is its radius of convergence. Then one can define a function $f : (-R, R) \to \mathbb{R}$ given by the formula

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \text{ whenever } |x| < R.$$

Thus the value of f at each point x is a convergent sum of real numbers.

It turns out that if we know this function f such that

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \text{ whenever } |x| < R,$$

then differentiation and integration work term-by-term. That is,

$$f'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1} \text{ whenever } |x| < R,$$

and

$$\int f(x)dx = \sum_{k=0}^{\infty} \frac{a_k x^{k+1}}{k+1} + C \text{ whenever } |x| < R.$$

Averages, Arc Length, Speed and Surface Area

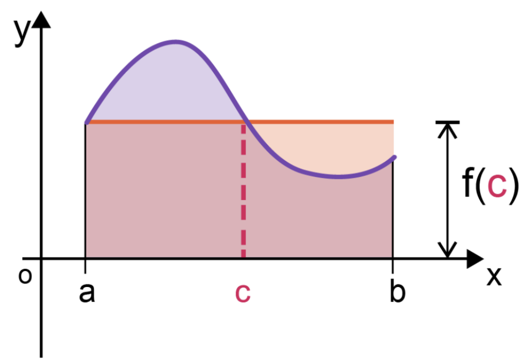
The average value of a function

The average value $\bar{f}$ of an integrable function f on a closed interval $[a, b]$ is defined by the formula

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

The mean value theorem for integrals. Suppose that f is continuous on $[a, b]$. Then there is a number c in (a, b) such that

$$\int_a^b f(t)\, dt = f(c)(b-a).$$



The arc length of a curve

Arc length for a parametrised curve

If C is described parametrically by

$$C = \{(x(t), y(t)) \in \mathbb{R}^2 : a \leq t \leq b\},$$

then its arc length ℓ is given by

$$\ell = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dx$$

Arc length for the graph of a function

If C is the graph

$$y = f(x), \quad x \in [a, b],$$

of a function f on $[a, b]$ then its arc length ℓ is given by

$$\ell = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Arc length for a polar curve

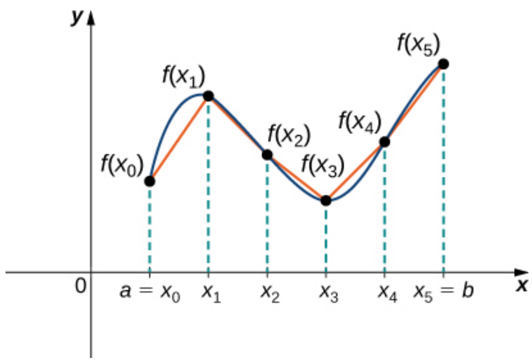
If C is described using polar coordinates by

$$r = f(\theta), \quad \theta_0 \leq \theta \leq \theta_1,$$

then its arc length ℓ is given by

$$\ell = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Visualisation of the arc length of a function



We can trace out the arc length of a function by joining nearby points on the curve and calculating the sum of the line segments. The arc length is essentially the limit as the distance between the points approaches 0.

The speed of a moving particle

The speed $v(t)$ of a particle P at time t is given by

$$v(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2},$$

where the functions x and y give the position $(x(t), y(t))$ of P at time t.

Surface area

Surface area for a parametrised curve

If C is described parametrically by

$$C = \{(x(t), y(t)) \in \mathbb{R}^2 : a \leq t \leq b\},$$

then the area A of the surface of revolution of the graph about the x-axis is given by

$$A = \int_a^b 2\pi y(t) \sqrt{[x(t)]^2 + [y(t)]^2}\, dx$$

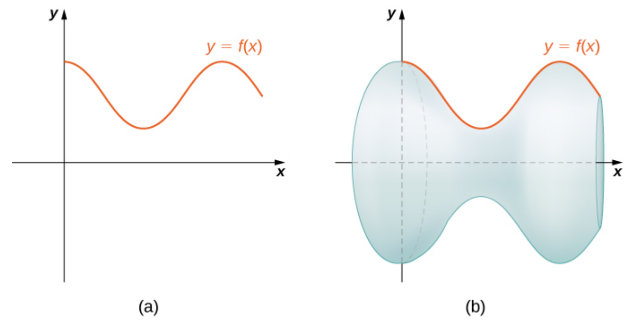
Surface area for the graph of a function

If C is the graph

$$y = f(x), \quad x \in [a, b],$$

of a function $f : [a, b] \to \mathbb{R}$, then the area A of the surface of revolution about the x-axis is given by

$$A = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx.$$



Surface area for a polar curve

If C is described using polar coordinates by

$$r = f(\theta), \quad \theta_0 \leq \theta \leq \theta_1,$$

then the area A of the surface of revolution about the x-axis is given by

$$A = \int_{\theta_0}^{\theta_1} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$