$\mathcal{A}$ccelerate $\mathcal{Q}$uestions *2022*

UNSW Mathematics Society

Question 1

Let $a > 1$ be a positive integer. Compute

$$\frac{\sum_{n=1}^{a^2-1} \sqrt{a+\sqrt{n}}}{\sum_{n=1}^{a^2-1} \sqrt{a-\sqrt{n}}}.$$

Write your answer to 3 decimal points.

Answer: $1+\sqrt{2} \approx 2.414$.

Proof. Define $A_n = \sqrt{a+\sqrt{n}}$ and $B_n = \sqrt{a-\sqrt{n}}$. Then we observe that

$$\begin{aligned}(A_n - B_n)^2 &= \left(\sqrt{a+\sqrt{n}} - \sqrt{a-\sqrt{n}}\right)^2 \\ &= a+\sqrt{n} - 2\sqrt{a+\sqrt{n}}\sqrt{a-\sqrt{n}} + (a-\sqrt{n}) \\ &= 2\left(a - \sqrt{a^2-n}\right).\end{aligned}$$

Therefore, we see that

$$A_n = B_n + \sqrt{2}\sqrt{a-\sqrt{a^2-n}}$$

and so

$$\sum_{n=1}^{a^2-1} A_n = \sum_{n=1}^{a^2-1} B_n + \sqrt{2}\sum_{n=1}^{a^2-1} \sqrt{a-\sqrt{a^2-n}}.$$

But it is easy to see that

$$\sum_{n=1}^{a^2-1} \sqrt{a-\sqrt{a^2-n}} = \sum_{n=1}^{a^2-1} \sqrt{a-\sqrt{n}} = \sum_{n=1}^{a^2-1} B_n.$$

Therefore,

$$\sum_{n=1}^{a^2-1} A_n = (1+\sqrt{2})\sum_{n=1}^{a^2-1} B_n \implies \frac{\sum_{n=1}^{a^2-1} A_n}{\sum_{n=1}^{a^2-1} B_n} = 1+\sqrt{2},$$

as required. □

Question 2

We say that a sequence a_n is 'good' if a_n is a sequence of 100 consecutive integers and contains 25 or more primes. Determine the number of good sequences.

Answer: 6.

Question 3

The sum of n positive integers is 19. What is the maximum possible product of these n numbers?

Answer: 972.

Proof. The goal here is to partition 19 and compute the product of the parts, and return the maximal product of such parts. To begin, for any $n \in \mathbb{N}$, let $x_1 + \cdots + x_n$ be a partition of 19 into n parts. We prove the first claim.

Claim. For any $1 \leq k \leq n$ where $x_k > 4$, then $x_k \leq 3(x_k - 3)$.

Proof. For simplicity and without loss of generality, we may assume that $x_1 \leq x_2 \leq \cdots \leq x_{n-1} \leq x_n$.

We proceed by induction. If $x_k = 5$, then

$$x_k = 5 \leq 6 = 3(x_k - 3).$$

Suppose that $x_k \leq 3(x_k - 3)$. Then define

$$x_{k+1} = \alpha + x_k$$

for some $\alpha \geq 0$. And so, we obtain

$$3(x_{k+1} - 3) \geq 3(\alpha + x_k - 3) = 3\alpha + 3(x_k - 3) \geq 3\alpha + x_k \geq x_{k+1}.$$

This proves the inductive step. □

Thus, by replacing $x_1 + \cdots + x_n$ with $3 + (x_1 - 3) + x_2 + \cdots + x_n$ will yield a bigger product. It, therefore, follows that the product-maximising partitions have parts no greater than 4.

However, $4 = 2 \times 2$, so we can just replace each 4 by two 2's. In other words, any product-maximising partition has parts strictly smaller than 4. Finally, if a partition contains only 2's and 3's, then observe that

$$2 \times 2 \times 2 = 8 < 9 = 3 \times 3.$$

In other words, our goal now is to extract as many 3's as we can and then all of the 2's will fill up the remaining spaces. Writing $19 = 3q + r$, we observe that $q = 5$ and $r = 4$. Thus, we should write 19 as the following partition

$$19 = (3 + 3 + 3 + 3 + 3) + (2 + 2).$$

The product of such a partition is

$$P = 3^5 \times 2^2 = 972.$$

□

Question 4

Given a, b and c are complex numbers, if $a+b+c = 4$, $a^2+b^2+c^2 = 20$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = -\frac{1}{8}$, find $a^5 + b^5 + c^5$.

Answer: 6.

Question 5

Find all integers x, y and z such that $x + y + z = x^3 + y^3 + z^3 = 3$.

Answer: 8.

Question 6

Compute

$$\lim_{n\to\infty} \cos^n\left(\sqrt{\frac{2}{n}}\right)$$

to 3 decimal points.

Answer: $0.368 \approx \frac{1}{e}$.

Question 7

Consider the cubic equation $x^3 - kx + (k+11) = 0$. Find all integer values of k for which the equation has at least 1 positive integer solution for x.

Answer: $k = 19, 25, 34, 59, 184$.

Question 8

Find the least integer $n > 1$ for which the sum of squares of consecutive integers from 1 to n is also a square number.

Answer: 24.

Question 9

Let $\{x\}$ denote the *fractional part* of x. A fractional part is defined as

$$\{x\} = \begin{cases} x - \lfloor x \rfloor, x \geq 0 \\ x - \lceil x \rceil, x < 0 \end{cases}$$

Compute

$$\int_0^1 \left\{\frac{1}{x} - \frac{1}{1-x}\right\} dx.$$

Answer: 1/2.

Proof. Let I be the integral we're after.

We first observe that, if x is not an integer, then $\{x\} + \{-x\} = 1$. Let $y = 1 - x$. Then we have that

$$\begin{aligned} I = \int_0^1 \left\{ \frac{1}{x} - \frac{1}{1-x} \right\} dx &= \int_0^1 \left\{ \frac{1}{1-y} - \frac{1}{y} \right\} dy \\ &= \int_0^1 \left\{ - \left(\frac{1}{y} - \frac{1}{1-y} \right) \right\} dy \\ &= 1 - \int_0^1 \left\{ \frac{1}{y} - \frac{1}{1-y} \right\} dy \\ &= 1 - I. \end{aligned}$$

Therefore, $I = 1/2$. □

Question 10

Let

$$J = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}.$$

Compute the value of J to 3 decimal points.

Answer: $1 - \ln 2 \approx 0.307$.

We provide two proofs of this.

Proof 1. Recall the Taylor series of $\ln(1+x)$; namely,

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$

Then

$$-\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1}$$

and so, substituting $x = 1$, we have

$$-\ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} = -1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}.$$

In other words,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} = 1 - \ln 2.$$

□

Proof 2. Observe that $\dfrac{(-1)^{n+1}}{n+1} = \displaystyle\int_0^1 (-1)^{n+1} x^n \, dx$. Therefore,

$$J = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} = \sum_{n=1}^{\infty} \int_0^1 (-1)^{n+1} x^n \, dx = \int_0^1 \sum_{n=1}^{\infty} (-1)^{n+1} x^n \, dx.$$

We now have a geometric series in disguise; observe that

$$\sum_{n=1}^{\infty} (-1)^{n+1} x^n = -\sum_{n=1}^{\infty} (-x)^n = \frac{x}{1+x}.$$

Therefore,

$$J = \int_0^1 \frac{x}{1+x} \, dx = \int_0^1 \left(1 - \frac{1}{1+x}\right) dx = 1 - \ln 2.$$

$\square$

Question 11

You are given a pile of coins. There are 73789 real coins that weigh the exact same. Then, there is one counterfeit mixed in. It is unknown whether it weighs slightly more or slightly less. You are given a scale. What is the minimum number of comparisons required to guarantee finding the counterfeit coin with the scales?

Answer: 12.

Question 12

Given that $f(x+y) = f(x) + f(y) + xy$ and $f(4) = 10$, find $f(2022)$.

Answer: 2045253.

Question 13

Rectangles are considered different if they vary in size or have different locations. How many different rectangles can be drawn along the lines of a chessboard? A chessboard is an 8×8 grid of squares.

Answer: 1296.

Question 14

Let ABCD be a trapezoid with bases 3 and 5, and equal legs 7. Find the length of the diagonal.

Answer: 8.

Question 15

A point is chosen at random inside the unit circle. Compute the probability that the point occurs at a distance less than $\frac{1}{3}$ from the centre to 3 decimal points.

Answer: $0.111 \approx \frac{1}{9}$.

Question 16

If $x^2 - x - 3 = 0$, find x^5 in terms of $ax + b$ where a and b are real constants. **Answer**: $19x + 12$.

Question 17

Find the number of 0s at the end of 999!. **Answer**: 244.

Question 18

Find the sum of all values of n which make $n^3 - 13n^2 + 45n - 33$ prime. **Answer**: 14.

Question 19

Find the greatest integer $n > 1$ for which $(n-1)! + 1 = n^2$. **Answer**: 5.

Question 20

Compute the number of digits in $2^{2022} - 1$. **Answer**: 609.