MATH1131/1141 Lab Test 1 Algebra Solutions to Samples

October 7, 2019

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Question 1

For the points $A(-9, 2, -9)$, $B(2, 0, 0)$ and $C(-2, -1, 1)$ you are given that

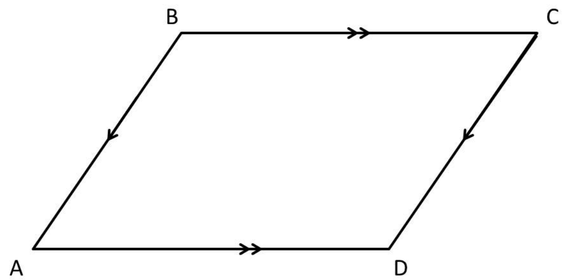
$$\overrightarrow{AB} = \begin{pmatrix} 11 \\ -2 \\ 9 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 7 \\ -3 \\ 8 \end{pmatrix}, \overrightarrow{BC} = \begin{pmatrix} -4 \\ -1 \\ -1 \end{pmatrix}$$

Find the position of vector $\overrightarrow{OD}$ of the point D such that the quadrilateral $ABCD$ (named in cyclic order) is a parallelogram. (Check the order carefully!)

Answer: We seek some vector in the parallelogram that involves D. Noting the cyclic order we can deduce that $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are diagonals.

By vector addition through the 'completing the parallelogram' technique, it is clear that

$$-\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{BD}.$$



Our aim is to isolate vector $\overrightarrow{OD}$, so we make $\mathbf{d}$ the subject

$$\begin{aligned}\mathbf{d} - \mathbf{b} &= -\overrightarrow{AB} + \overrightarrow{BC} \\ \mathbf{d} &= \mathbf{b} - \overrightarrow{AB} + \overrightarrow{BC} \\ &= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 11 \\ -2 \\ 9 \end{pmatrix} + \begin{pmatrix} -4 \\ -1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -13 \\ 1 \\ -10 \end{pmatrix}\end{aligned}$$

Hence our solution is $\overrightarrow{OD} = \begin{pmatrix} -13 \\ 1 \\ -10 \end{pmatrix}$

Question 2

Consider the three points $A(1, 5, -3), B(3, -3, 2), C(1, 15, 7)$.

Since the vector $\overrightarrow{AB}$ *is not* a scalar multiple of the vector $\overrightarrow{BC}$, we conclude that the points A, B and C are *not collinear*.

Answer: First we compute $\overrightarrow{AB}$ as required

$$\begin{aligned}\overrightarrow{AB} &= \mathbf{b} - \mathbf{a} \\ &= \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -8 \\ 5 \end{pmatrix}\end{aligned}$$

Now computing $\overrightarrow{BC}$

$$\begin{aligned}\overrightarrow{BC} &= \mathbf{c} - \mathbf{b} \\ &= \begin{pmatrix} 1 \\ 15 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 18 \\ 5 \end{pmatrix}\end{aligned}$$

Now: If 2 vectors $\mathbf{a}, \mathbf{b}$ are scalar multiples of each other then $\mathbf{a} = \lambda \mathbf{b}$ where $\lambda \in \mathbb{R}$. This however is not possible with the given set of vectors since the first components are negatives but the others are not. Therefore, the three points cannot be collinear.

Question 3

For the points $A(-6, 2, 1)$ and $B(-30, 5, -17)$, find a parametric equation of the straight line through A and B in the form

$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}$$

where $\mathbf{a}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$ and λ is a real parameter.
Does the point $C(-20, 8 - 33)$ lie on the straight line through A and B?

Answer:

We know by the definition of the parametric form, **a** corresponds to any vector that lies *on* the line. One option is the vector associated with the point A

$$\mathbf{a} = \begin{pmatrix} 0 \\ 5 \\ -6 \end{pmatrix}$$

Again by definition we know that **v** corresponds to a vector *parallel* to the line. One such vector is $\overrightarrow{AB}$

$$\begin{aligned} \overrightarrow{AB} &= \mathbf{b} - \mathbf{a} \\ &= \begin{pmatrix} -30 \\ 5 \\ -17 \end{pmatrix} - \begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix} \\ \mathbf{v} &= \begin{pmatrix} -24 \\ 3 \\ -18 \end{pmatrix} \end{aligned}$$

The point $C(-20, 8-33)$ does *not* lie on the through A and B

Suppose C did lie on the line:

$$\mathbf{x} = \begin{pmatrix} 0 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} -24 \\ 3 \\ 18 \end{pmatrix}$$

Then it follows that

$$\begin{pmatrix} -20 \\ 8 \\ -33 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} -24 \\ 3 \\ 18 \end{pmatrix}$$

By equating components we obtain the following system of equations

$$-20 = -24\lambda \quad (1)$$
$$8 = 3\lambda + 5 \quad (2)$$
$$-6 = 18\lambda - 6 \quad (3)$$

Observe that no value of λ will satisfy all 3 equations. For instance, $\lambda = 0$ satisfies (3) but fails (1) and (2). Hence we deduce that C cannot lie on the line.

Question 4

The line l in $\mathbb{R}^3$ given by the Cartesian equations

$$\frac{x-4}{-6} = \frac{y-6}{2} = \frac{z-2}{-4}$$

can be written in parametric vector form

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}$$

where $\mathbf{a}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$ and λ is a real parameter.

One possible choice for $\mathbf{a}$ is $\begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}$. Find a **different** choice for $\mathbf{a}$ and a vector $\mathbf{v}$ so that the vector equation above represents the line l.

Answer:

In this questions all we require is to convert from Cartesian to parametric form.

Let us follow the standard procedure by first introducing the desired parameter λ where

$$\lambda = \frac{x-4}{-6} = \frac{y-6}{2} = \frac{z-2}{-4}$$

Rearranging to make the components of $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ the subject.

$$x = 4 - 6\lambda$$
$$y = 6 + 2\lambda$$
$$z = 2 - 4\lambda$$

Thus our parametric form is given by:

$$\mathbf{x} = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix}$$

Note $\mathbf{x}$ is the locus of vectors which satisfy the equation of the line, with parameter λ. Thus, setting $\lambda = 1$, gives another possible vector which lies on the line i.e.

$$\mathbf{x} = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix} + \begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix}$$
$$\mathbf{x} = \begin{pmatrix} -2 \\ 8 \\ -2 \end{pmatrix}$$

Hence, a different choice for $\mathbf{a}$ is $\begin{pmatrix} -2 \\ 8 \\ -2 \end{pmatrix}$. It's clear that $\mathbf{v}$ can be given by $\begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix}$

Question 5

Consider the points $A(-2, 8, -4), B(-3, 9, -1)$, and $C(7, 2, -1)$. You are given that

$$\overrightarrow{AB} \times \overrightarrow{OB} = \begin{pmatrix} 28 \\ 10 \\ 6 \end{pmatrix}, \overrightarrow{OA} \times \overrightarrow{BC} = \begin{pmatrix} -28 \\ -40 \\ -66 \end{pmatrix} \text{ and } \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 21 \\ 30 \\ -3 \end{pmatrix}$$

The parallelopiped generated by the three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ has volume equal to?

Answer:

We want to find the volume of a parallelopiped so straight away the scalar triple product should come to mind.

Applying the formula $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$, we have

$$\left| \left(\begin{pmatrix} -2 \\ 8 \\ -4 \end{pmatrix} \times \begin{pmatrix} -3 \\ 9 \\ -1 \end{pmatrix} \right) \cdot \begin{pmatrix} 7 \\ 2 \\ -1 \end{pmatrix} \right| .$$

So, simplifying the cross product, we get

$$\begin{aligned} &= \left| \begin{pmatrix} 28 \\ 10 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 2 \\ -1 \end{pmatrix} \right| \\ &= |28 \times 7 + 10 \times 2 + 6 \times -1| \\ &= 221. \end{aligned}$$

Question 6

Consider the points $P(-4, 4, 0)$ and $Q(-6, 5, 1)$ and the vector

$$\mathbf{v} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} .$$

You are given that

$$\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} .$$

Let l be the straight line through $\mathbb{R}^3$ through the point P parallel to the vector $\mathbf{v}$.

The projection of $\overrightarrow{PQ}$ onto $\mathbf{v}$ is the vector $\text{proj}_{\mathbf{v}}\overrightarrow{PQ} = ?$

The shortest distance between the line l and the point Q is?

Answer:

To calculate the projection we simply apply the formula

$$\text{proj}_{\mathbf{v}}\overrightarrow{PQ} = \frac{\mathbf{v} \cdot \overrightarrow{PQ}}{|\mathbf{v}|^2}\mathbf{v}$$

This gives,

$$\text{proj}_{\mathbf{v}}\overrightarrow{PQ} = \frac{\begin{pmatrix}0\\3\\1\end{pmatrix} \cdot \begin{pmatrix}-2\\1\\1\end{pmatrix}}{\left|\begin{pmatrix}0\\3\\1\end{pmatrix}\right|^2}\begin{pmatrix}0\\3\\1\end{pmatrix}$$

Applying the dot product formula and further simplifying,

$$\begin{aligned}\text{proj}_{\mathbf{v}}\overrightarrow{PQ} &= \frac{4}{\left|\begin{pmatrix}0\\3\\1\end{pmatrix}\right|^2}\begin{pmatrix}0\\3\\1\end{pmatrix}\\ &= \frac{4}{10}\begin{pmatrix}0\\3\\1\end{pmatrix}\\ &= \begin{pmatrix}0\\\frac{6}{5}\\\frac{2}{5}\end{pmatrix}\end{aligned}$$

Recall the shortest distance between a point Q and a line l with parallel vector $\mathbf{v}$, is given by

$$\left|\overrightarrow{PQ} - \text{proj}_{\mathbf{v}}\overrightarrow{PQ}\right| = \left|\begin{pmatrix}-2\\1\\1\end{pmatrix} - \begin{pmatrix}0\\\frac{6}{5}\\\frac{2}{5}\end{pmatrix}\right| = \left|\begin{pmatrix}-2\\\frac{-1}{5}\\\frac{3}{5}\end{pmatrix}\right| = \sqrt{\frac{22}{5}}$$

Question 7

Consider the points $A(1,-2,-1), B(2,-5,-2)$, and $C(3,-5,-3)$. You are given,

$$\overrightarrow{AB} = \begin{pmatrix}1\\-3\\-1\end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix}2\\-3\\-2\end{pmatrix}$$

The projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$ is the vector $\text{proj}_{\overrightarrow{AC}}\overrightarrow{AB} =$?

The length of the altitude through B in the triangle ABC is?

Answer: Using the same formulae as above we compute the projection,

$$\text{proj}_{\overrightarrow{AC}}\overrightarrow{AB} = \frac{13}{17}\begin{pmatrix}2\\-3\\-2\end{pmatrix}$$

Calculating the length of the altitude is similar to finding the shortest distance in the previous question. B is our *point* and $\overrightarrow{AC}$ is the vector *parallel to the line*. Thus we have,

$$\begin{aligned}\text{Length of Altitude} &= \left|\overrightarrow{AB} - \text{proj}_{\overrightarrow{AC}}\overrightarrow{AB}\right| \\ &= \left|\begin{pmatrix}1\\-3\\-1\end{pmatrix} - \frac{13}{17}\begin{pmatrix}2\\-3\\-2\end{pmatrix}\right| \\ &= \left|\begin{pmatrix}\frac{-9}{17}\\\frac{-12}{17}\\\frac{9}{17}\end{pmatrix}\right| \\ &= \sqrt{\frac{18}{17}}\end{aligned}$$

Question 8

A plane in $\mathbb{R}^3$ is given in parametric vector form by the equation

$$\mathbf{x} = \begin{pmatrix}-9\\-5\\7\end{pmatrix} + \lambda\begin{pmatrix}-4\\1\\-7\end{pmatrix} + \mu\begin{pmatrix}9\\-3\\4\end{pmatrix}$$

Find a vector normal to the plane.

Answer: Recall, the cross product of two vectors yields a vector which is perpendicular to both vectors. Hence, the cross product of any two vectors parallel to the surface of a plane will give a normal vector.

From our understanding of the parametric form, we know that $\begin{pmatrix}-4\\1\\-7\end{pmatrix}$ and $\begin{pmatrix}9\\-3\\4\end{pmatrix}$ are vectors parallel to the plane

Therefore it follows that,

$$\begin{aligned}\text{Normal Vector } &= \begin{pmatrix}-4\\1\\-7\end{pmatrix} \times \begin{pmatrix}9\\-3\\4\end{pmatrix}\\ &= \begin{pmatrix}(1)\times(4)-(-7)\times(-3)\\(-7)\times(9)-(-4)\times(4)\\(-4)\times(-3)-(1)\times(9)\end{pmatrix}\\ &= \begin{pmatrix}-17\\-47\\3\end{pmatrix}\end{aligned}$$

Question 9

The plane in $\mathbb{R}^3$ with Cartesian equation

$$3x_1 - 6x_2 - x_3 = -2$$

can be expressed in parametric form

$$\mathbf{x} = \mathbf{a} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$$

where $\mathbf{a}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in $\mathbb{R}^3$ and λ_1 and λ_2 are real parameters. You are given that

$$\mathbf{v}_1 = \begin{pmatrix}2\\1\\0\end{pmatrix}$$

Find suitable vectors $\mathbf{a}$ and $\mathbf{v}_2$.

Answer: Another routine question, where we are required to convert from Cartesian to parametric from.

As per usual first we introduce the real parameters $\lambda_1 = x_1$ and $\lambda_2 = x_2$, leaving $x_3 = 3\lambda_1 - 6\lambda_2 + 2$.

Therefore our parametric equation becomes

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 3\lambda_1 - 6\lambda_2 + 2 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$$

A clear choice for a is $\mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

Now, **v1** and **v2** must be linearly independent to span a plane. So either $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$ are suitable vectors for $\mathbf{v_2}$ as neither are scalar multiples of $\mathbf{v_1}$

Question 10

The plane passing through the 3 points $A(2, -3, 1), B(4, -1, -2)$, and $C(-3, -2, -4)$ can be expressed as a parametric vector equation of the form

$$\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$$

where λ_1 and λ_2 are real parameters. You are given that

$$\mathbf{v}_1 = \begin{pmatrix} -7 \\ -1 \\ -2 \end{pmatrix}$$

Find suitable vectors $\mathbf{a}$ and $\mathbf{v}_2$.

Answer: $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent vectors *parallel* to the surface of the plane. Such vectors can be obtained by taking the difference of any 2 points on the plane and verifying their linear independence.
Setting $\mathbf{v}_2 = \mathbf{a} - \mathbf{b}$ yields

$$\begin{aligned}\mathbf{v}_2 &= \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -2 \\ 3 \end{pmatrix}\end{aligned}$$

Which is not a scalar multiple of $\mathbf{v}_1$ and thus the choice is verified as the set is linearly independent.

$\mathbf{a}$ corresponds to any point on the plane, a suitable choice is $\begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix}$

Question 11

Consider the line l given by the parametric vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$$

and the plane described by the Cartesian equation

$$3x + y + z = 2.$$

Find the value of the parameter λ at the point where the line and plane intersect

Answer: Since the question involves an intersection, we know straight away to solve the equations of the line and plane simultaneously.

Our goal is to find the parameter λ so we isolate the components of our parametric equation which are in terms of λ and substitute this into the Cartesian equation.

$$\begin{aligned} x &= -2\lambda \\ y &= -4-\lambda \\ z &= 2+3\lambda \end{aligned}$$

Substituting this into the cartesian equation of the plane

$$\begin{aligned} 3(-2\lambda)+(-4-\lambda)+(2+3\lambda) &= 2 \\ -6\lambda-\lambda+3\lambda-4+2 &= 2 \\ -4\lambda &= 4 \\ \lambda &= -1 \end{aligned}$$