MATH1081 Lab Test 2

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Question 1

a) Fill out the following table.

p	q	$\sim p \to q$	$p \vee \sim q$
T	T		
T	F		
F	T		
F	F		

b) The expression

$$\sim (p \vee \sim q) \wedge \sim (\sim p \to q)$$

is a

a) We shall fill the table and add an extra column that will help with part (b). To do these questions, we need to recall the outcome of the "implies" expression:

$$T \to T \implies T$$
$$T \to F \implies F$$
$$F \to T \implies T$$
$$F \to F \implies T$$

We also recall that

$$T \vee T \implies T \qquad T \wedge T \implies T$$
$$T \vee F \implies T \qquad T \wedge F \implies F$$
$$F \vee T \implies T \qquad F \wedge T \implies F$$
$$F \vee F \implies F \qquad F \wedge F \implies F$$

Using this knowledge, we can fill in our table

p	q	$\sim p \to q$	$p \vee \sim q$	$\sim (p \vee \sim q) \wedge \sim (\sim p \to q)$
T	T	T	T	F
T	F	T	T	F
F	T	T	F	F
F	F	F	T	F

b) A **tautology** occurs when, regardless of what p and q are, the statement is true. A **contradiction** occurs when the statement is always false, and a **contingent** statement would be dependent on the value of p and q.

By considering our table from part (a), we see that, regardless of what p and q are, the statement is always false. And hence, the expression given is a **contradiction**.

Question 2

The most common interpretation of the laws of Islam in Northern Africa is the Maliki school, founded in the 8th century AD.
With regards to inheritance the Maliki school says that an estate goes to the children so that sons get twice what the daughters get. Hermaphrodite children, whose gender is between male and female, get the average. Inheritance calculations for multiple hermaphrodite children require each individual permutation of genders to be considered.
Ancient texts that determine the inheritance for large numbers of hermaphrodite siblings are fairly common. These were probably contrived cases to demonstrate legal and mathematical prowess instead of actual inheritance cases. Demonstrate *your* mathematical prowess by calculating the inheritance problem below.
A parent dies and leaves 2 female daughters and 3 hermaphrodites. According to the Maliki school, what fraction of the estate goes to

- the female daughters?
- the hermaphrodites?

We shall begin by calculating how much of the estate goes to the female daughters. The problem, now, is that we don't know the gender of the hermaphrodites so we can't divide the estate according to the Maliki school. Instead, we shall take cases depending on the gender of the hermaphrodites.

Case 1. Hermaphrodites are all **male**.

This has a $\frac{1}{8}$ chance of happening since each of the hermaphrodites have 2 options and only one of which is male. To calculate the fraction of the estate in this case, we suppose that $M = 2F$ since the males get twice what the females get. Setting $M = 2F$, we can then find how much each female gets since the total estate can be divided in

$$\underbrace{(2F+2F+2F)}_{\text{male hermaphrodites}} + \underbrace{(F+F)}_{\text{female daughters}} = 8F.$$

So each female gets $\frac{1}{8}$ of the total estate. So for case 1, we have a total of

$$\frac{1}{8} \times 2 \times \frac{1}{8}$$

of the estate.

Case 2. Two of the hermaphrodites are **male** and one is **female**.

This has a $\frac{3}{8}$ chance of happening since the position of the female hermaphrodite will accumulate when calculating the fraction of the estate. To calculate the fraction of the estate, again, we shall set $M = 2F$, giving us

$$\underbrace{(2F+2F)+F}_{\text{hermaphrodites}} + \underbrace{(F+F)}_{\text{female daughters}} = 7F.$$

So each female daughter gets $1/7$ of the estate, and so this case gives us

$$\frac{3}{8} \times 2 \times \frac{1}{7}.$$

Case 3. Two of the hermaphrodites are **female** and one is male.

This has a $\frac{3}{8}$ chance of happening.

Again, we calculate the estate by setting $M = 2F$, which gives us

$$\underbrace{2F+(F+F)}_{\text{hermaphrodites}} + \underbrace{(F+F)}_{\text{female daughters}} = 6F.$$

Each female daughter gets $1/6$ of the estate, and so this case gives us

$$\frac{3}{8} \times 2 \times \frac{1}{6}.$$

Case 4. Hermaphrodites are all **female**.

Much like the case 1, this has a 1/8 chance of happening, with the estate divided as

$$\underbrace{(F+F+F)}_{\text{hermaphrodites}} + \underbrace{(F+F)}_{\text{female daughters}} = 5F.$$

So the female daughters share 1/5 of the estate each, and this case gives us

$$\frac{1}{8} \times 2 \times \frac{1}{5}.$$

So the female daughters' average fraction of the estate would just be the total fraction of each case added together

$$\left(\frac{1}{8} \times 2 \times \frac{1}{8}\right) + \left(\frac{3}{8} \times 2 \times \frac{1}{7}\right) + \left(\frac{3}{8} \times 2 \times \frac{1}{6}\right) + \left(\frac{1}{8} \times 2 \times \frac{1}{5}\right) = \frac{351}{1120}.$$

To calculate the hermaphrodites' average fraction of the estate, we take the complement to the female daughter's probability. This gives us

$$1 - \frac{351}{1120} = \frac{769}{1120}.$$

Question 3

a) For the production of a local play, 8 people auditioned and 6 of them joined the cast. Of these, 3 had named roles. In how many ways could this have happened?

b) In how many ways can 4 boys and 4 girls be arranged in a circle so that the boys and girls are alternating?

c) How many 4-digit numbers have the property that all of the digits are different, and all of the digits are odd?

a) Of the 8 people, we need to pick 6 of them to join the cast. This can be done in `comb(8, 6)` ways. And of these 6 people, we require 3 of them to have named roles. Since the positions of these named roles are distinguishable, then the ordering of these named roles matter. So we can arrange the named roles in `perm(6, 3)`, giving us a total of

```
comb(8, 6)*perm(6, 3)
```

ways.

b) We first fix a boy or girl into a position. Then to arrange the rest, we can imagine it as a straight line arrangement. It doesn't matter where the first boy sits but note that each of these positions do not change the number of arrangements since each initial seat is the "same". To arrange the rest (from the initial boy clockwise), consider the straight line arrangement

$$\underline{G} \quad \underline{B} \quad \underline{G} \quad \underline{B} \quad \underline{G} \quad \underline{B} \quad \underline{G}.$$

There is no alternative arrangement that ensures that the boys and girls are alternating. So to arrange this, we can arrange the girls in 4! ways and we can arrange the boys in 3! ways, giving us a total of

```
4!*3!
```

arrangements.

c) Since all of the digits are odd, we only have 5 choices (1, 3, 5, 7, 9). Since the ordering matters and the digits must be different, we have

```
perm(5, 4)
```

total arrangements.

If this isn't immediately clear, we can think about it using a straight line arrangement. Consider any 4-digit number. We have 5 choices for the first position and then 4 choices for the second position, 3 choices for the third position, and so on. This gives us

$$5 \times 4 \times 3 \times 2$$

total number of arrangements, which is equivalent to `perm(5, 4)` as we had before.

Question 4

A firm works 6 days a week. Every employee works 3 full days and 3 half days. A half day can be either morning or afternoon and two half-days cannot be on the same day.

a) How many possible different weekly schedules are there?

b) If the firm has 791 employees, how many people must have the same work schedule for a particular week?

b) What is the smallest number of employees needed to guarantee at least 6 workers have exactly the same schedule?

a) To calculate the total possible different schedules, we begin by selecting the full days. We have 6 days to choose the 3 full days. Since the ordering makes no difference, we can select the number of full days in $\binom{6}{3}$ = `comb(6, 3)` ways.

Then to select the remaining half-days, we have 3 days, each of which contains 2 slots. So for each of the days, we have 2 options and we want to choose 1 of them. This gives us $\left(\binom{2}{1}\right)^3$ = `2^3` ways of choosing the half-days. So in all, we have

```
comb(6, 3)*2^3
```

different schedules.

b) We shall apply the Pigeonhole principle with 791 pigeons and 160 pigeonholes. Note that 160 comes from our answer from part (a).

Applying the pigeonhole principle, we must have at least

$$\left\lceil \frac{791}{160} \right\rceil = 5$$

people sharing the same schedule. Note that the ceiling function will round the number up regardless of the decimal place. For example the ceiling of 3.14 is the same as the ceiling of 3.999, which is 4.

c) We can think about stacking each of the positions with a person who has a particular schedule. If we stack each position with 5 people, we would still not **guarantee** that 6 people share the same schedule. However, the next person will always give us one particular schedule with 6 people sharing the same schedule, so the smallest number would be

$$\mathbf{5} \times \mathbf{160} + \mathbf{1} = 801.$$

Question 5

a) A normal 12-sided die has each of the numbers 1 through to 12 written on its faces, and has an equal chance of landing on any one of these numbers.
If it is rolled 14 times, what is the probability that a 1 turns up exactly 9 times?

b) Suppose a normal 8-sided die is rolled 21 times. What is the probability that a 1 appears exactly 1 time, a 2 appears exactly 2 times, . . ., and a 6 appears exactly 6 times amongst these 21 rolls?

c) It is known that a 5-digit number does not contain any digits greater than 8. What is the probability that it is less than 76585?

a) This is a classic binomial probability question with n being the number of rolls and k being the number of times a 1 turns up. Using the binomial probability, we have

$$P = \binom{14}{9}\left(\frac{1}{12}\right)^9\left(\frac{11}{12}\right)^5,$$

which can be written in NUMBAS as `(comb(14, 9)*11^5)/12^4`.

b) Suppose that in 21 rolls, we had one 1, two 2's, three 3's, . . . six 6's. Then we can imagine this as a straight line arrangement with five different repeated elements. So the number of ways to arrange this is

$$\frac{21!}{\underbrace{1!2!3!4!5!6!}_{\text{repeated elements}}}.$$

So the probability of this occurring would be the result above over the total, which gives us

$$\frac{21!}{8^{21} \cdot (1!2!3!4!5!6!)},$$

which can be written as `21!/(8^21*1!*2!*3!*4!*5!*6!)`.

c) To begin, let's just label the number to be

$$\underline{A} \quad \underline{B} \quad \underline{C} \quad \underline{D} \quad \underline{E}.$$

We shall take cases for the value of A. If $A < 7$, then there are 6 choices for A since $A \neq 0$, followed by any one of the 9 digits that are less than 8. So if $A < 7$, we have 6×9^4 total arrangements.

If $A = 7$, then we shall take another two cases. If $B < 6$, then there exist 6 choices for B since we also include $B = 0$ in our choices, followed by any one of the 9 digits that are less than 8, which gives us 6×9^3 total arrangements.

If $B = 6$, then we take another two subcases. We repeat this process until we arrive at the desired number.

This gives us a total of

$$6 \times 9^4 + 6 \times 9^3 + 5 \times 9^2 + 8 \times 9^1 + 5 \times 9^0$$

number of arrangements, written as

```
6*9^4 + 6*9^3 + 5*9^2 + 8*9^1 + 5*9^0
```

in NUMBAS.

To find the final probability, we take the number of arrangements we have calculated and divide that by the total number of possible arrangements. To find the total number of arrangements, we consider all of the possible combinations that form the 5-digit number. The first number can contain numbers between 1 and 7 since the first number cannot be a 0. Hence, the first digit contains 7 different choices of digits. However, the remaining four digits can range from 0 to 7. Hence each of these positions will contain 8 different options. Thus, the total number of 5-digit numbers that can be made is 7×8^4. Thus, the probability that the 5-digit number is less than 76585 is

$$\frac{6 \times 9^4 + 6 \times 9^3 + 5 \times 9^2 + 8 \times 9^1 + 5 \times 9^0}{7 \times 8^4}.$$

Question 6

Consider the equation

$$x_1 + x_2 + \cdots + x_6 = 57,$$

where $x_1, x_2, \ldots, x_6 \in \mathbb{N}$.

How many solutions are there if:

a) $x_i \leq 10$ for all $1 \leq i \leq 6$?

b) $x_i \leq 19$ for all $1 \leq i \leq 6$?

c)

$$\begin{aligned} &x_1 \geq 10, \text{ and} \\ &x_i \equiv i \bmod 4 \quad \text{for all } 1 \leq i \leq 6? \end{aligned}$$

a) Let $y_i = 10 - x_i$. Since $x_i \leq 10$, then we see that y_i will be nonnegative. So we can frame this question to find the number of solutions for y_i. Rewriting each equation gives us $x_i = 10 - y_i$ and so our equation becomes

$$\begin{aligned} (10 - y_1) + (10 - y_2) + \cdots + (10 - y_6) &= 57 \\ y_1 + y_2 + y_3 + \cdots + y_6 &= 3. \end{aligned}$$

So the total number of solutions, using the stars and bars method, is

```
comb(8, 5).
```

b) If we apply the above method, we require that the new total must be ≤ 10. Applying this method in this case gives the right hand side a number that's bigger than 10, so we need to resort to another method.

We shall apply the principle of inclusion/exclusion. We shall consider the number of arrangements where one element is greater than 19, two elements is greater than 19, three elements, and so on.

Suppose that one element is greater than 19. Since it makes no difference whichever element we choose, we'll consider it for x_1. If $x_1 > 19$, then we can set $y_1 = x_1 - 20$, where y_1 is a nonnegative integer solution. Rearranging for x_1 gives us $x_1 = y_1 + 20$.

Then our equation becomes

$$\begin{aligned}(y_1 + 20) + x_2 + x_3 + \cdots + x_6 &= 57 \\ y_1 + x_2 + x_3 + \cdots + x_6 &= 37.\end{aligned}$$

Since y_1 is just another nonnegative integer and its solution is a corresponding solution to x_1, then y_1 makes no difference to x_1. Applying the stars and bars method of counting gives us the total number of ways of having one element > 19 to be

$$\binom{37+6-1}{6-1} = \binom{42}{5}.$$

The total number of ways this can happen is $\binom{6}{1}$.

Similarly, consider the case where two elements is greater than 19. For the sake of simplicity, consider $x_1, x_2 > 19$. Then following the same idea gives the new equation

$$y_1 + y_2 + x_3 + \cdots + x_6 = 57 - 20 - 20 \implies y_1 + y_2 + x_3 + \cdots + x_6 = 17.$$

Applying the stars and bars method of counting gives us

$$\binom{17+6-1}{6-1} = \binom{22}{5}.$$

Repeating this process again gives us a negative right hand side since 57 - 20 - 20 - 20 = -3. And so, we say there are no solutions. So we stop there and complete the question by applying the principle of inclusion/exclusion, which gives us

$$\underbrace{\binom{62}{5}}_{\text{total}} - \underbrace{\binom{6}{1}\binom{42}{5}}_{\text{one element}} + \underbrace{\binom{6}{2}\binom{22}{5}}_{\text{two elements}},$$

which can be written in NUMBAS as `comb(62, 5) - comb(6, 1)*comb(42, 5) + comb(6, 2)*comb(22, 5)`.

c) Let $x_i = 4y_i + i \bmod 4$. Then explicitly writing it out, we have the following

$$\begin{aligned}
x_1 &= 4y_1 + 1. \\
x_2 &= 4y_2 + 2. \\
x_3 &= 4y_3 + 3. \\
x_4 &= 4y_4. \\
x_5 &= 4y_5 + 1. \\
x_6 &= 4y_6 + 2.
\end{aligned}$$

Then our equation becomes

$$\begin{aligned}
&(4y_1 + 1) + (4y_2 + 2) + (4y_3 + 3) + 4y_4 + (4y_5 + 1) + (4y_6 + 2) = 57 \\
&\implies 4(y_1 + y_2 + y_3 + \cdots + y_6) + (1 + 2 + 3 + 1 + 2) = 57 \\
&\implies y_1 + y_2 + \cdots + y_6 = 12.
\end{aligned}$$

To satisfy the first condition, we require that $x_1 \geq 10$ so $4y_1 + 1 \geq 10$ and $y_1 > 2$. So letting $z_1 = y_1 - 3$, we have

$$z_1 + y_2 + \cdots + y_6 = 12 - 3 = 9.$$

Solving for z_1 and y_i using the stars and bars method gives us

$$\binom{9 + 6 - 1}{6 - 1} = \binom{14}{5}.$$

So our solution is simply `comb(14, 5)`.

Question 7

a) Find the solution to the recurrence relation

$$a_n = 9a_{n-1} - 8a_{n-2} \quad \text{for all } n \geq 2$$

which satisfies the initial conditions $a_0 = -8$ and $a_1 = -50$.

b) Find the general solution to the recurrence relation

$$b_n = 4b_{n-1} + 45b_{n-2} - 96n + 332 \quad \text{for all } n \geq 2$$

which satisfies the initial conditions $b_0 = 1$ and $b_1 = -7$.

c) Find the general solution to the recurrence relation

$$c_n = 7c_{n-1} - 12c_{n-2} + 4^n \quad \text{for all } n \geq 2$$

which satisfies the initial conditions $c_0 = 1$ and $c_1 = 11$.

a) We shall rewrite the equation so that one side contains all of the coefficients of a_k

$$a_n - 9a_{n-1} + 8a_{n-2} = 0.$$

This becomes a **homogeneous** equation since the right hand side is 0. Computing its characteristic equation, we arrive at

$$\lambda^2 - 9\lambda + 8 = 0.$$

So we see that the solutions to this characteristic equation are $\lambda = 8, \lambda = 1$.

So the general solution to the recurrence relation is

$$a_n = c_1 \cdot 8^n + c_2 \cdot 1^n.$$

To find the values for c_1 and c_2, we substitute our initial conditions and solve simultaneously for c_1 and c_2.

We have the equations

$$a_0 = c_1 + c_2 = -8,$$
$$a_1 = 8c_1 + c_2 = -50.$$

Solving simultaneously, we arrive at $c_1 = -6, c_2 = -2$.

So the general solution to the recurrence relation is

$$a_n = \texttt{-6*8^n - 2*1^n}.$$

b) Again, we shall rewrite the equation so that one side contains all of the terms with b_k

$$b_n - 4b_{n-1} - 45b_{n-2} = -96n + 332.$$

This is now a **nonhomogeneous** equation since the right hand side is no longer 0. So we expect our general solution to be a combination of the homogeneous solution h_n and the particular solution p_n.

To solve these types of equations, we first solve its homogeneous equation

$$b_n - 4b_{n-1} - 45b_{n-2} = 0.$$

Computing its characteristic equation, we arrive at

$$\lambda^2 - 4\lambda - 45 = 0 \implies \lambda = 9, \quad \lambda = -5.$$

So the homogeneous solution would be

$$h_n = c_1 \cdot 9^n + c_2 \cdot (-5)^n.$$

Now, since the right hand side is a linear equation, we guess our particular solution to be

$$p_n = an + b.$$

Now this is a solution we picked specifically so that

$$p_n - 4p_{n-1} - 45p_{n-2} = -96n + 332.$$

We can expand and simplify the left hand side to arrive at

$$\begin{aligned} p_n - 4p_{n-1} - 45p_{n-2} &= (an + b) - 4(a(n-1) + b) - 45(a(n-2) + b) \\ &= n(a - 4a - 45a) + (b + 4a - 4b + 90a - 45b) \\ &= (-48a)n + (94a - 48b). \end{aligned}$$

Comparing our coefficients, we arrive at $a = 2$, $b = -3$.

So we have

$$p_n = 2n - 3.$$

Our general solution then becomes

$$b_n = h_n + p_n = c_1 \cdot 9^n + c_2 \cdot (-5)^n + (2n - 3).$$

To find c_1 and c_2, we substitute our initial conditions with the final solution being

$$b_n = 9^n + 3(-5)^n + 2n - 3,$$

which can be written as

$b_n =$ `9^n + 3*(-5)^n + 2*n - 3`.

c) As with the previous questions, we shall rewrite the equation to the form

$$c_n - 7c_{n-1} + 12c_{n-2} = 4^n.$$

This is a **nonhomogeneous** equation so we should expect the final solution to be a combination of its homogeneous h_n and its particular solution p_n.

To find the homogeneous equation, we compute the characteristic equation

$$\lambda^2 - 7\lambda + 12 = 0.$$

This has roots at $\lambda = 3, \quad \lambda = 4$. So the homogeneous equation becomes

$$h_n = c_1 \cdot 3^n + c_2 4^n.$$

The problem now comes because our right handside is a solution in our homogeneous equation. If we took $p_n = a4^n$, then we have the recurrence relation to be 0, not 4^n.

Instead, take $p_n = an4^n$. If that is in the homogeneous equation, multiply p_n by n until you arrive at a form not in the homogeneous equation.

Setting $p_n = an4^n$, we should expect

$$p_n - 7p_{n-1} + 12p_{n-2} = 4^n.$$

Expanding and simplifying our left side, we arrive at

$$n(16a - 28a + 12a) + (28a - 24a) = 16 \implies a = 4.$$

So our particular solution is

$$p_n = 4n \cdot 4^n.$$

So our general solution is

$$c_n = c_1 \cdot 3^n + c_2 \cdot 4^n + 4n \cdot 4^n.$$

To find our values for c_1 and c_2, we substitute our initial conditions and solve simultaneously for c_1 and c_2. We arrive at

$$c_1 = 9, \quad c_2 = -8.$$

So our general solution is

$$c_n = 9 \cdot 3^n - 8 \cdot 4^n + 4n \cdot 4^n,$$

which can be written as

$$c_n = \texttt{9*3^n - 8*4^n + 4n*4^n}.$$

