MATH1081 Lab Test 1

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at hello@unswmathsoc.org, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Question 1

In a class of 37 students:

- 17 study German
- 20 study Physics,
- 10 study both Biology and German,
- 13 study both Biology and Physics,
- 9 study both German and Physics,
- 7 study all three subjects, and
- 5 study none of these subjects.

a) How many students study Biology?

b) Writing B, G, and P for the sets of students studying Biology, German, and Physics respectively, evaluate $|(P^c \cup G)^c \cup B|$.

Solutions

a) **Answer**: 20

By considering the inclusion exclusion principle, we have that

$$|B \cup G \cup P| = |B| + |G| + |P| - |B \cap G| - |B \cap P| - |P \cap G| + |B \cap P \cap G|.$$

This gives us $37 - 5 = 17 + 20 + |B| - 10 - 13 - 9 + 7$ so $|B| = 20$.

b) **Answer**: 25

We have

$$\begin{aligned}
|(P^c \cup G)^c \cup B)| &= |(P \cap G^c) \cup B| \qquad \text{(De Morgan's law)} \\
&= |(P \cap G^c) \cap B^c| + |B| \\
&\text{(using inclusion-exclusion principle)} \\
&= |P| + |P \cap B \cap G| - |P \cap G| - |P \cap B| + |B| \\
&= 5 + 20 \\
&= 25.
\end{aligned}$$

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Question 2

For any integer k, let S_k be the set defined by:

$$S_k = \left\{ n \in \mathbb{Z} \mid 2k + 3 \leq n \leq \frac{3}{2}k + 10 \right\}.$$

a) What is $S_3 - S_1$?

b) i. Find $|\mathcal{P}(S_1 \times S_3)|$.

ii. Find $|\mathcal{P}(S_1) \times \mathcal{P}(S_3)|$.

c) i. Find $|\mathcal{P}(S_3) - \mathcal{P}(S_1)|$.

ii. Find $|\mathcal{P}(S_3) \cup \mathcal{P}(S_1)|$.

iii. Find $|\mathcal{P}(S_3) \cap \mathcal{P}(S_1)|$.

Note: There will likely only be one (b) and one (c) type question in the actual test.

Solutions

a) **Answer**: `set`(12,13,14)

From the provided rule we can determine that the set S_3 is the list of integers from 9 to 14 and the set S_1 is the set of integers from 5 to 11. The set $S_3 - S_1$ contains the elements that are in S_3 but not S_1 which is the integers from 12 to 14.

b) i. **Answer**: 2^{42}.

Recall that $|\mathcal{P}(A)| = 2^{|A|}$. Therefore, we will compute $|S_1 \times S_3|$. This is equivalent to $|S_1| \times |S_3|$. We have that $|S_1| = |\{5, \ldots, 11\}| = 7$ and $|S_3| = |\{9, \ldots, 14\}| = 6$. Therefore, we have that

$$|\mathcal{P}(S_1 \times S_3)| = 2^{|S_1 \times S_3|} = 2^{6 \times 7} = 2^{42}.$$

ii. **Answer**: 2^{13}.

Recall that $|A \times B| = |A| \times |B|$. Therefore, we have that

$$|\mathcal{P}(S_1) \times \mathcal{P}(S_3)| = |\mathcal{P}(S_1)| \times |\mathcal{P}(S_3)| = 2^{|S_1|} \times 2^{|S_3|} = 2^{6+7} = 2^{13}.$$

c) i. **Answer**: $2^6 - 2^3$.

Recall that $A - B = A \cap B^c$. We can read this as *the set of elements in A that do not belong in B*. Therefore, to compute the number of such elements, we can think of A as the *universe set*. Then we are excluding all of the elements from B which belong in our *universe set*. This gives us the following expression

$$|A - B| = |A| - |A \cap B|.$$

To see a more formal proof of this result, see the appendix.

In our example, we are computing

$$|\mathcal{P}(S_3) - \mathcal{P}(S_1)| = |\mathcal{P}(S_3)| - |\mathcal{P}(S_3) \cap \mathcal{P}(S_1)|.$$

To compute $|\mathcal{P}(S_3) \cap \mathcal{P}(S_1)|$, we firstly see that

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B).$$

To see why, you may want to visit the appendix. Therefore, it suffices to then compute $|S_3 \cap S_1|$. Recall that S_1 is the set $\{5, \ldots, 11\}$ and S_3 is the set $\{9, \ldots, 14\}$. The elements that are common to both are $S_1 \cap S_3 = \{9, 10, 11\}$ and so, $|S_1 \cap S_3| = 3$. Therefore, we have that

$$|\mathcal{P}(S_1) \cap \mathcal{P}(S_3)| = |\mathcal{P}(S_1 \cap S_3)| = 2^{|S_1 \cap S_3|} = 2^3.$$

This gives us

$$|\mathcal{P}(S_3) - \mathcal{P}(S_1)| = |\mathcal{P}(S_3)| - |\mathcal{P}(S_1) \cap \mathcal{P}(S_3)| = 2^6 - 2^3.$$

ii. **Answer**: $2^6 + 2^7 - 2^3$.

Using the inclusion/exclusion principle, we have that

$$|\mathcal{P}(S_3) \cup \mathcal{P}(S_1)| = |\mathcal{P}(S_3)| + |\mathcal{P}(S_1)| - |\mathcal{P}(S_1) \cap \mathcal{P}(S_3)|.$$

We have computed every component earlier, so this is just plugging and chugging.

iii. **Answer**: 2^3. See c), part i.

Question 3

a) Consider the function

$$f : \mathbb{R}_0^+ \to \mathbb{R}_0^+, f(x) = x(x+3)+2$$

Complete the following to make a true statement:

Since the equation $f(x) =$ [ ] has
[no/exactly one/more than one solution(s)],
we conclude that f is [injective/surjective/not injective/not surjective].

b) Consider the function

$$g : \mathbb{R} \to \mathbb{R}_0^+, g(x) = x^2.$$

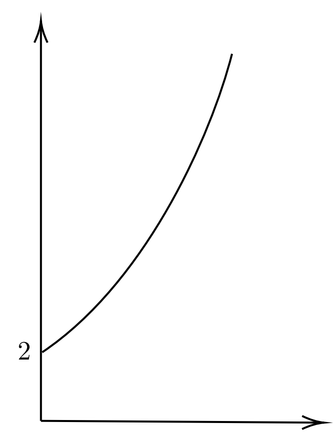
Complete the following to make a true statement:

Since the equation $g(x) =$ [ ] has
[no/exactly one/more than one solution(s)],
we conclude that g is [injective/surjective/not injective/not surjective].

Solutions

a) **Answer**: 1 (not only possible answer), no solutions, not surjective.

We can see from the graph of $f(x)$ below that the function f is not surjective since the codomain $\mathbb{R}_0^+$ and range $[2, \infty)$ are not equivalent. Since there is no x such that $f(x) \in [0, 2)$, any number in the interval from $[0, 2)$ will suffice for the first blank.



b) **Answer**: 1 (not only answer), more than one solution, not injective

In this case, g is surjective but is clearly not injective (one-to-one). Any real number *except* 0 will suffice for the first blank (since $g(x) = 0$ has only one solution, i.e. $x = 0$).

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Question 4

Suppose $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and that the function $f : S \to S$ is given by:

$$f(x) = \left(3x^2 + 5x + 4\right) \bmod 10.$$

Let $T = \{4, 6\}$.

Recall that the Numbas syntax for the set $\{a, b, c\}$ is $\texttt{set}(a, b, c)$.

a) What is $f(T)$?

b) What is $f^{-1}(T)$?

c) Complete the sentence:

f is ______

Solutions

We have the following outcomes:

- $f(0) = 4$
- $f(1) = 2$
- $f(2) = 6$
- $f(3) = 6$
- $f(4) = 2$
- $f(5) = 4$
- $f(6) = 2$
- $f(7) = 6$
- $f(8) = 6$
- $f(9) = 2$

a) **Answer**: $\texttt{set}(2)$
See red, clearly $f(\{4, 6\}) = \{2\}$.

b) **Answer**: $\texttt{set}(0, 2, 3, 5, 7, 8)$
See blue. The pre-image is given by $f^{-1}(T) = \{s \in S : f(s) \in T\}$, i.e. the values in S that obtain values in T when f is applied.

c) **Answer**: neither injective nor surjective.
Firstly, f is not injective (one-to-one), in fact, it is many-to-one.

Secondly, f is not surjective since the range does not equal the co-domain i.e. there exists elements of the co-domain that are not reached by the function of any element of the domain.

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Question 5

Two positive integers x and y are chosen, and their GCD and LCM are found to be the following:
$\texttt{gcd}(x, y) = 1470 = 2 \times 3 \times 5 \times 7^2$, and
$\texttt{lcm}(x, y) = 560290500 = 2^2 \times 3^3 \times 5^3 \times 7^3 \times 11^2$.

a) You are told that $x \neq \texttt{lcm}(x, y)$.
Given only this information, what is the largest possible value of x ?

b) You are now told that $x = 26460 = 2^2 \times 3^3 \times 5 \times 7^2$.
What is the value of y ?

To answer this question we must understand how the gcd and lcm contribute to the possibilities of x and y. The most important thing to note is that for every prime factor in the prime factorisation of the gcd and lcm, the gcd displays the minimum index of that factor between x and y, while the lcm displays the maximum index of that factor.

Given that we are only working with 2 numbers, to achieve these maximum and minimum indices they must be distributed across x and y, such that for any prime factor a, x contains the maximum index and y contains the minimum index of this factor or vice versa. This is how we can intuit that the product of the gcd and lcm is equal to the product of x and y.

a) $2 \times 3^3 \times 5^3 \times 7^3 \times 11^2$
By considering the distribution of factors, to make x the largest we provide it the largest indexes of each prime factor. to make it not equal to the lcm, we can then inspect the smallest difference in prime factor indexes between the gcd and lcm to have x, which was originally our lcm, decrease by the smallest amount.

b) $2 \times 3 \times 5^3 \times 7^3 \times 11^2$
using the fact that the product of the lcm and the gcd is equal to the product of x and y we solve this equation for y.

Question 6

When evaluating a number modulo m, be sure to give your answer in its lowest non-negative form, that is, as an element of $\{0, 1, 2, \ldots, m-1\}$.

a) Evaluate 7^{283} mod 11.

b) Evaluate 7^{118} mod 70.

Solutions

These questions apply Fermat's little theorem: if p is a prime number, then

$$a^p \equiv a \pmod{p}.$$

Furthermore, if a and p are coprime, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

a) **Answer**: 2

Set $a = 7$ and $p = 11$ then the second result gives

$$7^{10} \equiv 1 \pmod{11}$$

$$\text{(raising both sides to the 28th power)}$$

$$7^{280} \equiv 1 \pmod{11}$$

$$7^{283} \equiv 7^3 \pmod{11}$$

$$\therefore 7^{283} \equiv 2 \pmod{11}.$$

b) **Answer**: 49

We can break this question down into multiple modular reductions since $70 = 2 \times 5 \times 7$. Following a similar process as above, we get that

$$7^{118} \equiv 1 \pmod{2},\ 7^{118} \equiv 4 \pmod{5} \text{ and } 7^{118} \equiv 0 \pmod{7}.$$

By inspection, 49 satisfies these results and so $7^{118} \equiv 49 \pmod{70}$.

(By inspection here involves looking at the multiples of 7 and checking their remainders when divided 2 and 5.)

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Question 7

Solve each of the following modular arithmetic equations, giving your answer as a set of all possible solutions in the given modulus.
- If there are no solutions, enter `set()`.
- If there is one solution, say 1 , enter `set(1)`.
- If there are multiple solutions, say 1 and 2 , enter `set(1,2)`.

When evaluating in modulo m, give each answer in its lowest non-negative form - that is, as an element of $\{0, 1, 2, \ldots, m-1\}$.

a) Solve $598x \equiv 4 \pmod{2507}$.
b) Solve $20x \equiv 5 \pmod{53}$.
c) Solve $608x \equiv 20 \pmod{1708}$.
d) Solve $130x \equiv 20 \pmod{365}$.

The following results can all be determined through the process of the Extended Euclidean Algorithm, determining the gcd and then working backwards to determine the multiple.

a) `set()`
There do not exist any integer solutions x that satisfy the equation, as $\gcd(598, 2507) = 23$, which does not divide 4.

b) `set(40)`
Applying the Extended Euclidean Algorithm, we get $8 \times 20 + (-3) \times 53 = 1$. Multiplying both sides by 5, we get $40 \times 20 + (-15) \times 53 = 5$. As $\gcd(20, 53) = 1$, there is only one solution in the given modulus, which is $x = 40$.

c) `set(295, 722, 1149, 1576)`
Applying the Extended Euclidean Algorithm, we get $59 \times 608 + (-21) \times 1708 = 4$. Multiplying both sides by 5, we get $295 \times 608 + (-105) \times 1708 = 20$. As $\gcd(608, 1708) = 4$, there are four solutions in the given modulus, which are given by the set $\{295, 295 + \frac{1708}{4}, 295 + 2 \times \frac{1708}{4}, 295 + 3 \times \frac{1708}{4}\}$, or $\{295, 722, 1149, 1576\}$.

d) `set(17, 90, 163, 236, 309)`
As 5 is a factor of 130, 20 and 365, we can simplify the equation to become $26x \equiv 4 \pmod{73}$. Applying the Extended Euclidean Algorithm, we get $(-14) \times 26 + 5 \times 73 = 1$. Multiplying both sides by 4, we get $(-56) \times 26 + 20 \times 73 = 4$.

As $\gcd(26, 73) = 1$, there is only one solution in the modulus of 73, which is $x = 40$. However, the given modulus in the question is 365, so we can generate the other solutions of x by adding 73 until just before exceeding 365, thus giving us the set $\{17, 90, 163, 236, 309\}$.

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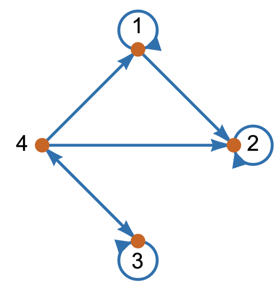
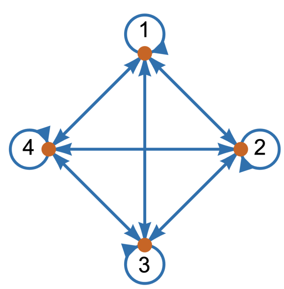
Question 8

For each of the arrow diagrams below, indicate whether they represent reflexive, symmetric and/or transitive relations.
Marks will be deducted for each incorrect selection but the minimum possible total mark for this question is 0.

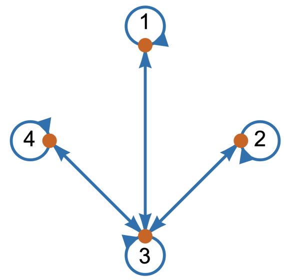
a) This graph is not reflexive, as 4 does not map to itself. It is not symmetric as 1 maps to 2 but 2 does not map back to 1. It is not transitive, as 4 maps to 3 then back to 4, but there does not exist a direct mapping of 4 to itself.



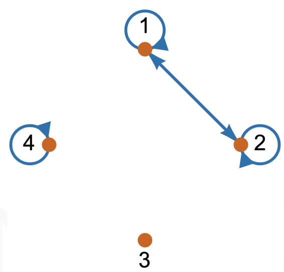
b) This graph is reflexive, as all four numbers map to themselves. It is symmetric as all the arrows are double sided. It is transitive, as from all four numbers, it is possible to reach any number directly.



c) This graph is reflexive, as all four numbers map to themselves. It is symmetric as all the arrows are double sided. It is not transitive, as 1 maps to 3 then to 2, but there does not exist a direct mapping of 1 to 2.



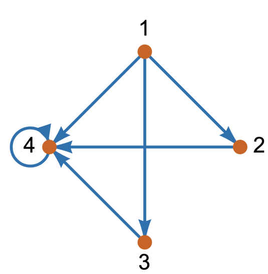
d) This graph is not reflexive, as 3 does not map to itself. It is symmetric as all the arrows are double sided. It is transitive as 4 only maps to 4, 3 is completely isolated, and all possible mappings exist between 1 and 2.



e) This graph is not reflexive, as 1, 2 and 3 do not map to themselves. It is not symmetric as 1 maps to 2 but 2 does not map back to 1. As for transitivity, the only non-trivial three-step mappings are:

- 1 maps to 2, and 2 maps to 4;
- 1 maps to 3, and 3 maps to 4;

and 1 indeed maps to 4, proving that the relation is transitive.



Appendix

Proposition. *Let A and B be finite sets. Then $|A - B| = |A| - |A \cap B|$.*

Proof. Firstly, we see that

$$(A \cap B) \cup (A \cap B^c) = A \cap (B \cup B^c) = A.$$

Now, suppose that $x \in (A \cap B) \cap (A \cap B^c)$. Since $x \in A \cap B$, we have that $x \in B$. Similar,y since $x \in A \cap B^c$, we have that $x \in B^c$. This is a contradiction and so,

$$(A \cap B) \cap (A \cap B^c) = \emptyset.$$

This implies that $(A \cap B)$ and $(A \cap B^c)$ is the *disjoint union* of A. Therefore,

$$|A| = |(A \cap B) \cup (A \cap B^c)| = |A \cap B| + |A \cap B^c|.$$

Rearranging the equation proves the result. □

Proposition. *Let A and B be finite sets. Then $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.*

Proof. To prove a set equality, we prove inclusions in two directions.

- ($\subseteq$). Let $X \in \mathcal{P}(A \cap B)$. Then $X \subseteq A \cap B$. This implies that $X \subseteq A$ and $X \subseteq B$, which implies that $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$ (i.e. $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$). This proves the forward subset containment.
- ($\supseteq$). Let $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$. But this implies that $X \subseteq A$ and $X \subseteq B$, which implies that $X \subseteq A \cap B$. But this implies that $X \in \mathcal{P}(A \cap B)$. This proves the reverse subset containment.

Thus, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$. □